

THE DOUBLE OF A HYPERBOLIC MANIFOLD AND NON-POSITIVELY CURVED EXOTIC PL STRUCTURES

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ABSTRACT. We give examples of non-compact finite volume real hyperbolic manifolds of dimension greater than five, such that their doubles admit at least three non-equivalent smoothable PL structures, two of which admit a Riemannian metric of non-positive curvature while the third does not. We also prove that the doubles of non-compact finite volume real hyperbolic manifolds of dimension greater than four are differentiably rigid.

0. INTRODUCTION

Some of the most interesting aspects of the relationship between geometry and topology are the rigidity theorems. Among the most well known are, for instance, Bieberbach's Theorem: two homotopically equivalent flat manifolds are affinely isomorphic (hence diffeomorphic), or Mostow's Theorem: two homotopically equivalent closed hyperbolic n -manifolds are isometric (hence diffeomorphic), for $n > 2$, or Farrell-Jones' Topological Rigidity Theorem: two homotopically equivalent closed n -manifolds, one of which is non-positively curved, are homeomorphic, for $n > 4$ ("curvature" means "sectional curvature"). On the other hand, Farrell and Jones also gave counterexamples to "non-positive curvature implies differentiable rigidity". In fact, they gave counterexamples to "strictly negative curvature implies differentiable rigidity" (known as the Lawson-Yau conjecture), in dimensions larger than six (see [7]). In dimension six, counterexamples appear in [15], and in dimensions less than six, the question remains open.

Thus we have that non-positive curvature implies topological ($n > 4$) but not differentiable ($n > 5$) rigidity. Also, the results in [15] and [10] show that non-positive curvature (in fact, even negative curvature) does not imply PL rigidity either. That is, there are examples of negatively curved closed Riemannian n -manifolds, $n > 5$, that are homeomorphic but not PL homeomorphic.

On the other hand, there is a simple example of an n -manifold, $n > 4$, with many differentiable structures (some of them inducing different PL structures) such that only one differentiable structure admits a non-positively curved Riemannian metric. This manifold is the n -torus, and the result just mentioned follows from the torus theorem of Lawson and Yau [13] and Gromoll and Wolf [11].

Now, if M is one of the examples mentioned above (i.e., M has different negatively curved PL structures), we have that $M \times T^m$ has many different PL structures, some non-positively curved and some not: we can change the PL structure

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on M so that we get different non-positively curved PL structures on $M \times T^m$; if we change the PL structure on T^m , it is not difficult to see that we get PL structures that do not admit non-positively curved PL structures. Still, these examples are too artificial, because there is too much zero curvature. In this paper, we get better examples with much less zero curvature. For this we study the double of a non-compact finite volume real hyperbolic manifold.

A non-compact finite volume real hyperbolic n -manifold is homeomorphic to the interior of a compact manifold with boundary. Each boundary component is a flat manifold of dimension $n - 1$. The compact manifold with boundary can be doubled along its boundary to form DM , the *double* of M . It is well known that the hyperbolic metric can be modified on each cusp (see section 1) to produce a metric of non-positive curvature. Here we construct examples of non-compact finite volume real hyperbolic manifolds of dimension greater than five, such that their doubles admit at least three non-equivalent smoothable PL structures, two of which admit a Riemannian metric of non-positive curvature while the other does not.

Explicitly, here is the statement of our main theorem:

Theorem 1. *There are examples of non-compact finite volume real hyperbolic n -manifolds M , $n > 5$, such that*

- (i) *DM has, at least, three non-equivalent (smoothable) PL structures Σ_1 , Σ_2 , Σ_3 .*
 - (ii) *Σ_1 and Σ_2 admit a Riemannian metric with non-positive curvature, and negative curvature outside a hypersurface.*
 - (iii) *Σ_3 does not admit a Riemannian metric with non-positive curvature.*
- In fact, Σ_3 does not even admit a piecewise flat metric of non-positive curvature.*

At this point we think it is interesting to mention a result, which is of independent interest, that we needed during the proof of the main theorem (to show that non-concordant PL structures are, in fact, not PL equivalent, see corollary 9.6). This result tells us that the doubles of non-compact finite volume real hyperbolic manifolds are differentially rigid.

Theorem 2. *Let $f : DM \rightarrow DM$ be a homeomorphism, where DM is the double of a non-compact finite volume real hyperbolic n -manifold M , $n \geq 3$. Then f is homotopic to a diffeomorphism $g : DM \rightarrow DM$.*

The proof of this theorem actually proves also:

Theorem 2A. *Let $f : DM_1 \rightarrow DM_2$ be a homotopy equivalence, where DM_1 and DM_2 are the doubles of non-compact finite volume real hyperbolic n -manifolds, $n \geq 5$. Then f is homotopic to a diffeomorphism $g : DM_1 \rightarrow DM_2$.*

The proof of Theorem 2A uses Mostow's Rigidity Theorem and results of Farrell and Jones. We now return to the discussion of our main theorem. Theorem 1 follows from Theorems 3, 4, 5 and 6 below. Theorem 3 gives conditions for a construction of doubles with non-equivalent smoothable PL structures that admit Riemannian metrics of non-positive curvature. Also, Theorem 4 gives conditions for a construction of doubles with PL structures that do not admit a Riemannian metric of non-positive curvature.

Theorem 3. *Let M be a non-compact finite volume hyperbolic n -manifold, $n > 5$, with a large cusp C_M diffeomorphic to $T^{n-1} \times (0, \infty)$. Assume either*

- (1) n is odd, or
 (2) n is even and there is a totally geodesic hypersurface $N \subset M$, with exactly one cusp contained in the cusp C_M of M .

Then the double DM of M admits (at least) two different (smoothable) PL structures admitting Riemannian metrics with non-positive curvature.

For the definition of large cusp see 2.12, and to understand what we mean by “a cusp contained in a cusp” see 4.3.

Theorem 4. *Let M, N be two non-compact finite volume real hyperbolic manifolds such that*

- (1) N is a totally geodesic submanifold of M ,
 (2) $\dim M = n \geq 5$, $\dim N = n - 3$,
 (3) the normal bundle of N is trivial,
 (4) M has a cusp C_M diffeomorphic to $T^{n-1} \times (0, \infty)$,
 (5) N has exactly one cusp contained in C_M .

Then the double DM of M has a differentiable structure \mathcal{E} , such that

- (i) (DM, \mathcal{E}) is not PL equivalent to DM , and
 (ii) (DM, \mathcal{E}) does not admit a Riemannian metric with non-positive curvature.

In fact, DM does not even admit a piecewise flat metric with non-positive curvature.

Note that in both theorems we demand the non-compact finite volume real hyperbolic n -manifold M to have a cusp diffeomorphic to $T^{n-1} \times (0, \infty)$ (the existence of such manifolds is granted by theorem 5 below).

The idea of the proof of theorem 3 is to reglue the cusp (the one diffeomorphic to $T^{n-1} \times (0, \infty)$) using an exotic diffeomorphism of T^{n-1} . If the cusps are “large”, the smooth structure obtained by this regluing admits a Riemannian metric of non-positive curvature. This metric agrees with the original hyperbolic metric on a complement of a collared neighborhood of T^{n-1} . To prove that the PL structure induced by the smooth structure obtained by the regluing is not PL equivalent to the original one, we just have to prove, by a corollary of theorem 2 (see corollary 9.6) that this PL structure is not concordant to the original one. But recall that, by results of Kirby and Siebenmann, the set of concordant classes of PL structures on a PL manifold of dimension greater than four is in bijective correspondence with its third cohomology group with coefficients in \mathbb{Z}_2 . In view of this, it is not difficult to prove (see section 3) that there is a bijection between the cohomology classes corresponding to the PL structures obtained by this type of regluing and the image of the natural map $\iota_3 : H^3(DM, DM \setminus T^{n-1}) \rightarrow H^3(DM)$. Hence we need $\iota_3 \neq 0$. It turns out (see lemma 4.4 and remark 4.5) that ι_3 is never zero for n odd. If n is even $\iota_3 \neq 0$ if there is a codimension one, totally geodesic submanifold N such that the given cusp of M “contains exactly one cusp of N ”. Note that these two conditions are exactly (1) and (2) of the statement of theorem 3.

The idea of the proof of theorem 4 is the following. Let N be a totally geodesic submanifold of codimension 3 in M with trivial normal bundle. Suppose that N intersects the cross section of the given cusp in a homologically non-trivial fashion. Let $x \in H^3(DM)$ be the cohomology class dual to $[DN] \in H_{n-3}(DM)$. Then we will prove that it is a consequence of the Flat Torus Theorem that the PL structure corresponding to x does not admit a Riemannian metric of non-positive curvature.

It remains to prove that there are non-compact finite volume real hyperbolic manifolds with the desired properties. For this we use a standard method, following ideas of J. J. Millson. Indeed, let \mathcal{Q} be a rational quadratic form of signature $(n, 1)$, $n \geq 5$. Then, for certain integers a , the pair (\mathcal{Q}, a) determines a non-compact finite volume real hyperbolic n -manifold, $n \geq 5$, that we denote by $M(\mathcal{Q}, a)$ (these manifolds are arithmetically defined real hyperbolic manifolds; see section 7).

For every integer b , $M(\mathcal{Q}, ba)$ is a finite cover of $M(\mathcal{Q}, a)$.

The next theorem says that these arithmetically defined non-compact finite volume real hyperbolic manifolds have finite covers with cusps having tori as cross sections:

Theorem 5. *If $M(\mathcal{Q}, a)$ is as above, then it admits finite covers $M(\mathcal{Q}, ba)$ having one cusp diffeomorphic to $T^{n-1} \times (0, \infty)$.*

In fact, $M(\mathcal{Q}, a)$ admits finite covers $M(\mathcal{Q}, ca)$ with all cusps diffeomorphic to $T^{n-1} \times (0, \infty)$ (corollary 7.3). Also, there is a corollary of the proof of theorem 5 which says that, by taking further congruence subgroups (i.e., $M(\mathcal{Q}, qba)$, for some q), we can assume the cusp to be large.

Finally, we prove that we can find manifolds satisfying conditions (1) or (2) of theorem 3, and (1)–(5) of theorem 4. We will require that the quadratic form \mathcal{Q} satisfies:

0.1. \mathcal{Q} has the form $\mathcal{Q}(x_1, \dots, x_{n+1}) = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2 - \lambda_{n+1} x_{n+1}^2$, with $\lambda_i > 0$, $\lambda_i \in \mathbb{Q}$, and $\lambda_1 = \lambda_2 = \lambda_3$. Also, if $n \leq 6$, $\lambda_4 x_4^2 + \dots + \lambda_n x_n^2 - \lambda_{n+1} x_{n+1}^2$ has a rational isotropic vector.

Theorem 6. *Let $M(\mathcal{Q}, a)$ be as before with \mathcal{Q} satisfying (0.1). Then $M(\mathcal{Q}, a)$ admits finite covers M_0 , such that there are non-compact finite volume real hyperbolic manifolds M_1, M_2 satisfying the following conditions:*

- (1) $\dim M_0 = n \geq 5$, $\dim M_1 = n - 1$, $\dim M_2 = n - 3$;
- (2) $M_2 \subset M_1 \subset M_0$ and the inclusions are totally geodesic;
- (3) the normal bundle of M_2 in M_0 is trivial;
- (4) M_0 has a large cusp C_{M_0} diffeomorphic to $T^{n-1} \times (0, \infty)$;
- (5) M_i has exactly one cusp contained in C_{M_0} , $i = 1, 2$.

The finite covers obtained are not necessarily of the form $M(\mathcal{Q}, a)$.

We remark that in [1] examples are given of doubles of non-compact finite volume hyperbolic manifolds with exotic differentiable structures not admitting a non-positively curved metric. In these examples the PL structure does not change.

For other related results see [9], where examples are given of non-compact finite volume hyperbolic manifolds with exotic differentiable structures not admitting a pinched negatively curved metric.

The paper is structured as follows. In section 1 we define the double of a non-compact finite volume hyperbolic manifold. In section 2 we give some geometric lemmas that will allow us to provide exotic triangulations with a non-positively curved metric. In this section we also discuss large cusps. In section 3 we state some triangulation lemmas that we will use to change triangulations. Section 4 has more lemmas and remarks. In section 5 we prove theorem 3. In section 6 we prove theorem 4. Theorem 5 is proven in section 7 and theorem 6 in section 8. We prove theorem 2 in section 9. Finally, we have an appendix in which we prove a technical lemma.

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1. THE DOUBLE

1.1. Let M be a non-compact finite volume real hyperbolic manifold. Then it has a finite number of cusps, each one isometric to some $F_i \times [0, \infty)$ with metric $e^{-2t}S_i + dt^2$, where the F_i 's are compact connected manifolds with flat metric S_i (see [2], sec. D3). In what follows we identify the cusps with $F_i \times [0, \infty)$. The double DM of M is obtained by cutting M in the cusps, that is, consider $M \setminus \{F_i \times (b, \infty)\}$ for some positive real number b , and glue two copies along the boundary $F_i \times \{b\}$, for all i , by the identity map. DM has a unique (up to diffeomorphism) differentiable structure such that its restriction to each copy of $M \setminus \{F_i \times (b, \infty)\}$ is the differentiable structure induced by the hyperbolic structure. This is the canonical differentiable structure on DM .

Note that DM can be given a non-positively curved metric in the following way. Given $\epsilon > 0$, it is straightforward to find an $f_\epsilon : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $f_\epsilon'' \geq 0$, $f_\epsilon(t) = e^{t+b}$ for $t \geq \epsilon$ and $f_\epsilon(t) = e^{-t+b}$ for $t \leq -\epsilon$. Then if $N_i = F_i \times (-c, c)$, $c > \epsilon$, with metric $f_\epsilon^2 S_i + dt^2$, we have that N_i is non-positively curved. To see this, note that $f_\epsilon^2 S_i + dt^2$ is a warped metric (see [3] p. 23) so that the sectional curvature for the plane at the point $(p, t) \in F_i \times (-c, c)$ generated by the orthonormal basis $\{\lambda(\frac{\partial}{\partial t})_t + v_p, w_p\}$, where $v_p, w_p \in T_p F_i$, is given by

$$(1.2) \quad -\frac{f_\epsilon''}{f_\epsilon} \lambda^2 - \frac{(f_\epsilon')^2}{f_\epsilon^2} \|v\|^2$$

(see [3], p. 27). So, because each S_i is flat and $f_\epsilon'' \geq 0$, we have that N_i with metric $f_\epsilon^2 S_i + dt^2$ is non-positively curved. Then $F_i \times (-c, -\epsilon), F_i \times (\epsilon, c) \subset N_i$ are both isometric to $F_i \times (b - (\epsilon + c), b - \epsilon) \subset M \subset DM$; so we can put near $F_i = F_i \times \{b\} \subset DM$ the metric of N_i ($F_i \times \{b\} \subset DM$ corresponds to $F_i \times \{0\} \subset N_i$), obtaining in this way a non-positively curved metric on DM . Note that we can arrange for the curvature to be zero just in the $F_i = F_i \times \{b\} \subset DM$, and by making ϵ small we have curvature -1 outside a small neighborhood of the $F_i \times \{b\} \subset DM$.

2. GEOMETRIC LEMMAS

Let M be an n -dimensional compact manifold and $I = [a, b]$, $0 \leq a \leq b$, a closed interval. Then the tangent space of $M \times I$ at (x, t) is isomorphic to $T_x M \oplus \mathbb{R}(\frac{\partial}{\partial t}|_t)$. Also, denote by $p^i, i = 1, 2$, the projection of $M \times I$ onto M and I , respectively.

Here we see that if we have a metric on the model $M \times I$, satisfying certain conditions, then, provided that we stretch and enlarge it enough, we have control over the warped metric. We also need to prove a second lemma, because when we try to fit the model on some manifold, we have to take coverings (to make it large), but these coverings make the model grow at different rates in different directions.

2.1. Now consider Riemannian metrics A on $M \times I$ with the following properties:

(a) For any $v \in T_x M$, $A(v, \frac{\partial}{\partial t}) = 0$.

(b) $A(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1$.

Equivalently, $A = S_t + dt^2$, where S_t is a metric on M depending on t .

2.2. For any metric A satisfying (2.1), and a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ (write α and t for the variables of f and f_α for $f(\alpha, t)$) define a new metric A_α on $M \times I$, depending on α :

$$A_\alpha(v, w) = f_\alpha^2(\alpha t) A_1(v, w) + \alpha^2 A_2(v, w) \quad v, w \in T_{(x,t)} M \times I$$

where $A_i(v, w) = A(p_*^i(v), p_*^i(w)) = (p^i)^* A(v, w)$.

Equivalently, if $A = S_t + dt^2$, then $A_\alpha = f_\alpha^2(\alpha t) S_t + \alpha^2 dt^2$.

2.3. As before, let f be such that for $t \in I$:

- (a) $f_\alpha(\alpha t) \rightarrow \infty$, as $\alpha \rightarrow \infty$;
- (b) $(\frac{f'_\alpha}{f_\alpha})(\alpha t) \rightarrow \ell_1 < \infty$, as $\alpha \rightarrow \infty$;
- (c) $(\frac{f''_\alpha}{f_\alpha})(\alpha t) \rightarrow \ell_2 < \infty$, as $\alpha \rightarrow \infty$.

We will require that these limits are uniform for t on a bounded interval.

Write $k_0 = \max\{\ell_1^2, \ell_2\}$, and $k_1 = \min\{\ell_1^2, \ell_2\}$.

2.4. Lemma. *Let A be a Riemannian metric on $M \times I$ satisfying (2.1) and f as in (2.3). Then given $\epsilon > 0$, there is an L such that, for $\alpha > L$,*

$$-k_0 - \epsilon < K(A_\alpha) < -k_1 + \epsilon,$$

i.e., all sectional curvatures of A_α lie in the interval $(-k_0 - \epsilon, -k_1 + \epsilon)$.

The proof is the same as lemma 3.5 of [7], but there is a proof in the appendix, because we will need some details afterwards.

2.5. Note that the constant L of the lemma depends only on C (see (a.2) of the appendix); so if A' is any other metric on $M \times I$ satisfying (2.1) and such that (a.2) is true for some $C' \leq C$, then the lemma holds for A' with the same L .

2.6. When M is the n -torus T^n , we can say a little more than lemma 2.4, but first some definitions.

If M is parallelizable with $X = (X_1, \dots, X_n)$, n linearly independent globally defined vector fields, then any Riemannian metric A expressed in this basis induces a smooth function $A_X : M \rightarrow \mathcal{P}$, where \mathcal{P} is the space of symmetric positive definite n by n matrices. Now given any $\varphi \in GL(n, \mathbb{R})$ (this is the space of invertible n by n real matrices), we can change A to a metric A^φ , by defining (note that the definition depends on X)

2.7. $A_X^\varphi = \varphi^T \cdot A_X \cdot \varphi$ (the T stands for transpose).

2.8. Now when $M = T^{n-1} = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$, consider $X' = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\}$ and $X = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial t}\}$, the canonical globally defined vector fields on T^{n-1} and $T^{n-1} \times I$, respectively. Note that these vector fields come from locally defined coordinate systems (x_1, \dots, x_n, t) (they are the projection of the canonical coordinates of $\mathbb{R}^n = \mathbb{R}^{n-1} \times [0, 1]$).

2.9. Let $\varphi \in GL(n-1, \mathbb{R}) \subset GL(n, \mathbb{R})$, where the last inclusion is induced by the inclusion $\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ given by $(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, t)$. Multiplication by φ induces a linear map $x \mapsto \varphi x$ and we use the same letter to denote this map. Suppose that φ is distance non-decreasing, that is $|\varphi(v)| \geq |v|, \forall v \in \mathbb{R}^{n-1}$ (or simply $|\varphi| \geq 1$), where the bars denote the Euclidean norm. Note that if $A = S_t + dt^2$ is a metric on $T^{n-1} \times I$ satisfying (2.1) and φ is as before, then

$A^\varphi = S_t^\varphi + dt^2$ (rigorously, $A_X^\varphi = (S_t)_{X'}^\varphi \subset GL(n-1, \mathbb{R}) \subset GL(n, \mathbb{R})$, where X' and X are as in (2.8)).

2.10. Lemma. *Let A be a Riemannian metric on $T^{n-1} \times I$ satisfying (2.1) and f as in (2.3). Then given $\epsilon > 0$, there is an L such that, for $\alpha > L$, and φ distance non-decreasing,*

$$-k_0 - \epsilon < K(A_\alpha^\varphi) < -k_1 + \epsilon.$$

Remark. The point here is that the constant L is independent of φ , provided it is distance non-decreasing.

Proof. The proof is the same as the proof of lemma 2.4 (see the appendix), except that we have to find a constant C , as in (a.2), that is, independent of φ satisfying (2.9).

Let $A_X = (g_{ij})$ be the representation of A in the basis X , where X is as in (2.8). Now, $A_X(x, t)$ is positive definite; thus, it can be written as $E^T(x, t).E(x, t)$, for some matrix E depending on (x, t) . Note that E is smooth (and globally defined because the bundle $O(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}) \xrightarrow{\eta} \mathcal{P}$, $\eta(B) = B^T B$, is trivial) in (x, t) . Because $T^{n-1} \times I$ is compact, we can find a constant C_1 such that

$$\left| \frac{\partial^{k+s} g_{ij}}{\partial x_{i_1} \dots \partial x_{i_k} \partial t^s} \right| < C_1, \quad |E_{ij}^{-1}(x, t)| < C_1, \quad \forall i, j, \quad k + s \leq 2.$$

For each $(x_0, t_0) \in T^n \times I$, define new coordinates near (x_0, t_0) as follows:

$$\begin{aligned} \bar{x} &= E(x_0, t_0). \varphi.x, \\ \bar{t} &= t \end{aligned}$$

where $x = (x_1, \dots, x_n)$ are coordinates as in (2.8) (without the t variable). Also, write $\bar{x}_0 = E(x_0, t_0). \varphi.x_0$ and $\bar{t}_0 = t_0$.

Then if (\bar{g}_{ij}^φ) is A^φ in (\bar{x}, \bar{t}) -coordinates (that is, $A_X^\varphi = (\bar{g}_{ij}^\varphi)$), we have

$$\begin{aligned} (\bar{g}_{ij}^\varphi)(\bar{x}, \bar{t}) &= [\varphi^{-1}.E^{-1}(x_0, t_0)]^T \cdot [\varphi^T.A_X(x, t).\varphi] \cdot [\varphi^{-1}.E^{-1}(x_0, t_0)] \\ &= E^{-T}(x_0, t_0).A_X(x, t).E(x_0, t_0), \end{aligned}$$

that is, it is independent of φ . Because $A_X(x, t) = E^T(x, t).E(x, t)$, we have that $(\bar{g}_{ij}^\varphi)(\bar{x}_0, \bar{t}_0)$ is the identity matrix, so that (b) of (a.2) is true. To complete the proof, we have to show that (c) of (a.2) holds. By the chain rule,

$$\left(\frac{\partial \bar{g}_{ij}^\varphi}{\partial \bar{x}_1}, \dots, \frac{\partial \bar{g}_{ij}^\varphi}{\partial \bar{x}_n} \right) = \left[\left(\frac{\partial \bar{g}_{ij}^\varphi}{\partial x_1}, \dots, \frac{\partial \bar{g}_{ij}^\varphi}{\partial x_n} \right) \right] \cdot [(\varphi)^{-1}.E^{-1}(x_0, t_0)]$$

and second derivatives

$$\left(\frac{\partial^2 \bar{g}_{ij}^\varphi}{\partial \bar{x}_k \partial \bar{x}_l} \right) = \left(\frac{\partial^2 \bar{g}_{ij}^\varphi}{\partial x_k \partial x_l} \right) \cdot [(\varphi)^{-1}.E^{-1}(x_0, t_0)]^2.$$

But φ is distance non-decreasing (i.e., $|\varphi| \geq 1$). Then φ^{-1} is distance non-increasing (i.e., $|\varphi^{-1}| \leq 1$), so that

$$\begin{aligned} \left| \left(\frac{\partial \bar{g}_{ij}^\varphi}{\partial \bar{x}_1}, \dots, \frac{\partial \bar{g}_{ij}^\varphi}{\partial \bar{x}_n} \right) \right| &\leq |\varphi^{-1}| \cdot |E^{-1}(x_0, t_0)| \cdot \left| \left(\frac{\partial \bar{g}_{ij}^\varphi}{\partial x_1}, \dots, \frac{\partial \bar{g}_{ij}^\varphi}{\partial x_n} \right) \right| \\ &\leq |E^{-1}(x_0, t_0)| \cdot \left| \left(\frac{\partial \bar{g}_{ij}^\varphi}{\partial x_1}, \dots, \frac{\partial \bar{g}_{ij}^\varphi}{\partial x_n} \right) \right|, \end{aligned}$$

and for the second derivatives,

$$\begin{aligned} \left| \left(\frac{\partial^2 \bar{g}_{ij}^\varphi}{\partial \bar{x}_k \partial \bar{x}_l} \right) \right| &\leq \left| \left(\frac{\partial^2 \bar{g}_{ij}^\varphi}{\partial x_k \partial x_l} \right) \right| \cdot |E^{-1}(x_0, t_0)|^2 \cdot |(\varphi)^{-1}|^2 \\ &\leq \left| \left(\frac{\partial^2 \bar{g}_{ij}^\varphi}{\partial x_k \partial x_l} \right) \right| \cdot |E^{-1}(x_0, t_0)|^2. \end{aligned}$$

Thus we can find a constant C as in (c) of (a.2) because (\bar{g}_{ij}^φ) is independent of φ , and its derivatives in \bar{x} -coordinates can be majorated by its derivatives in x -coordinates, that do not depend on φ either. This completes the proof of the lemma.

2.11. Denote by I the canonical flat metric on $T^{n-1} = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$. That is, I written in the basis $X' = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\}$ (see 2.6, 2.8) is just the identity matrix.

Then, for $\varphi \in GL(n-1, \mathbb{R})$, $I_{X'}^\varphi = \varphi^T \varphi$ (see 2.7).

2.12. We say that the torus $T^{n-1} = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ with metric I^φ is *large*, if $(\frac{1}{c_n})\varphi$ is distance non-decreasing, where c_n is a constant (that depends only on the dimension n) that will be determined on the proof of theorem 3 (see 5.1). A flat torus is large if it is isometric to $\mathbb{S}^1 \times \dots \times \mathbb{S}^1$ with metric I^φ , I^φ large. Note that every flat torus is isometric to \mathbb{R}^{n-1}/L , with $L \subset \mathbb{R}^{n-1}$ a lattice generated by some vectors v_1, \dots, v_{n-1} . Hence, this flat torus is isometric to $\mathbb{S}^1 \times \dots \times \mathbb{S}^1$ with metric I^φ , where $\varphi = [v_1 \dots v_{n-1}]$. Then the flat torus is large iff we can find a basis v_1, \dots, v_{n-1} of L such that $|\varphi| \geq 1$, for $\varphi = [v_1 \dots v_{n-1}]$.

A cusp of the form $T^{n-1} \times [0, \infty)$ with metric $e^{-2t}S + dt^2$ is large if T^{n-1} with metric S is large.

2.13. We say that a finite cover $p : T^{n-1} \rightarrow T^{n-1}$ corresponding to a subgroup of finite index H of $\pi_1(T^{n-1}) \cong \mathbb{Z}^{n-1}$ is simple with respect to a basis $\mathcal{B} = \{v_1, \dots, v_{n-1}\}$ of $\pi_1(T^{n-1})$, if H has a basis of the form $\{m_1 v_1, \dots, m_{n-1} v_{n-1}\}$, $m_i \in \mathbb{Z}$. The minimum length of the simple cover is just $\min\{|m_i|\}$.

2.14. Lemma. Fix a base $\mathcal{B} = \{v_1, \dots, v_{n-1}\}$ of $\pi_1(T^{n-1})$, and $S = I^\varphi$ a (flat) metric on T^{n-1} . Then there is a K such that if $p : T^{n-1} \rightarrow T^{n-1}$ is a simple cover (with respect to \mathcal{B}) of minimum length $\geq K$, then T^{n-1} with metric p^*S is large.

Proof. Consider T^{n-1} with canonical coordinates $x = (x_1, \dots, x_{n-1})$, i.e., these coordinates come from \mathbb{R}^{n-1} by the covering projection $\mathbb{R}^{n-1} \rightarrow T^{n-1}$. We identify $\pi_1(T^{n-1})$ with $\mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$. Thus $v_i \in \mathbb{Z}^{n-1}$. Now, if p is simple, then $p : T^{n-1} \rightarrow T^{n-1}$ is such that $p(v_i) = m_i v_i$, for some $m_i \in \mathbb{Z}$. Let B be the matrix whose columns are the v_i 's. Then p written in the basis v_1, \dots, v_{n-1} is given by $d = \text{diag}\{m_1 \dots m_{n-1}\}$ and it follows that $p(x) = BdB^{-1}x$, for $x \in \mathbb{R}^{n-1}$. Consequently, the pullback of the flat metric $S = I^\varphi$ in T^{n-1} , written in the canonical basis $\{e_i\}$ is

$$(BdB^{-1})^T \varphi^T \varphi (BdB^{-1}) = (\varphi BdB^{-1})^T (\varphi BdB^{-1}).$$

It is now easy to see that if the length of p (i.e., $\min\{|m_i|\}$) is large, we get $(\frac{1}{c_n})\varphi BdB^{-1}$ distance non-decreasing. In fact, it is enough to take $K \geq c_n |B| |B^{-1}| |\varphi^{-1}|$. This completes the proof.

3. TRIANGULATION LEMMAS

Recall that if M is a PL manifold and $C \subset M$, a closed subset (assume $m = \dim M \geq 6$ or $\dim M \geq 5$ and $\partial M \subset C$), then there is a one-to-one correspondence between $\check{H}^3(M, C; \mathbb{Z}_2)$ (this is Čech cohomology) and the set of concordance classes of PL structures on M that agree with the given one on a neighborhood of C . We can choose this correspondence to be such that it sends the given PL structure to 0.

Given a concordance class of triangulations $[\tau]$, by $c_{[\tau]} = c_\tau \in \check{H}^3(M, C; \mathbb{Z}_2)$, we denote the corresponding cohomology class; also, given a cohomology class c , we write $[\tau_c] = [\tau]_c$ for the corresponding concordance class of triangulations.

We have the following

3.1. Lemma. *Let $p : \tilde{M} \rightarrow M$ be a covering, $C \subset M$ closed and $m = \dim M \geq 6$ (or $\dim M \geq 5$ and $\partial M \subset C$). Suppose M has a PL structure τ_0 and denote by $\tilde{\tau}_0$ the pullback $p^*\tau_0$ of τ_0 and make these two triangulations correspond to zero in $\check{H}^3(\tilde{M}, p^{-1}(C); \mathbb{Z}_2)$ and $\check{H}^3(M, C; \mathbb{Z}_2)$, respectively. Then $[\tau]_{p^*c} = [p^*\tau_c]$ for all $c \in \check{H}^3(M, C; \mathbb{Z}_2)$. Equivalently, $c_{p^*\tau} = p^*c_\tau$ for every PL structure τ on M .*

Note that if τ_1 and τ_2 are concordant PL structures on M , then $p^*\tau_1$ and $p^*\tau_2$ are also concordant.

The proof of this lemma appears in [15].

Now, given a PL manifold M , we show how to change PL structures by cutting along a closed hypersurface N of M , and gluing back with a twist.

Denote by M_χ the $CAT(= PL \text{ or } DIFF)$ manifold obtained by cutting along N (a CAT closed hypersurface), and identifying by χ the two copies of N so obtained, where N is a CAT closed hypersurface and $\chi : N \rightarrow N$ is a CAT isomorphism. In what follows, we assume that the relative set is nice enough (for example, a deformation retract of a subcomplex) so that we replace Čech cohomology by singular cohomology.

3.2. Lemma. *Let M be a PL orientable n -manifold, $n \geq 6$, N a closed PL hypersurface with a tubular neighborhood $g : W \cong_{PL} N \times [-1, 1]$ of N in M , where $g(N) = N \times \{0\}$, and $J \subset N$ open with \bar{J} compact. Then for every $c \in H^3(M, M \setminus J; \mathbb{Z}_2)$, there is a PL isomorphism $\chi : N \rightarrow N$, such that M_χ (that is, its PL structure) corresponds to c (by the correspondence that sends the given PL structure to 0) and χ is the identity outside a compact neighborhood of \bar{J} .*

The proof of this lemma is given in [15].

3.3. Remark. Note that if τ is smoothable, then, using the differentiable s -cobordism theorem, we can choose χ to be smooth (see [15]).

4. MORE LEMMAS AND REMARKS

First some remarks.

4.1. Let M^n be hyperbolic, non-compact with finite volume and $F^{n-1} \times [0, \infty)$ one of its cusps (recall that we are identifying the cusp with $F^{n-1} \times [0, \infty)$). Then if $\iota : F^{n-1} \hookrightarrow F^{n-1} \times [0, \infty) \subset M$ is the inclusion, $\iota_* : \pi_1(F^{n-1}) \rightarrow \pi_1(M)$ is one-to-one. This is because (see 1.2) we can put a non-positively curved metric on

the double DM of M , such that F^{n-1} is a totally geodesic flat submanifold of DM , and then the map

$$\pi_1(F^{n-1}) \rightarrow \pi_1(M) \rightarrow \pi_1(DM)$$

is one-to-one, which means that ι_* is one-to-one.

4.2. We claim that $T^{n-1} \times [0, 3\alpha]$ with metric $e^{-2t}I^\varphi + dt^2$ (see 2.11) is isometric to $T^{n-1} \times [0, 3]$ with metric $I_\alpha^\psi + \alpha^2 dt^2$, where $\psi = (e^{-4\alpha})\varphi$ and the warping function is $f(\alpha, t) = e^{4\alpha - \alpha t}$. To see this, we have $I_\alpha^\psi + \alpha^2 dt^2 = f(\alpha, t)^2 I^{(e^{-4\alpha})\varphi} + \alpha^2 dt^2 = e^{8\alpha - 2\alpha t} [(e^{-4\alpha})\varphi]^T [(e^{-4\alpha})\varphi] + \alpha^2 dt^2 = (e^{-2\alpha t})\varphi^T \varphi + \alpha^2 dt^2$, and note that $T^{n-1} \times [0, 3]$ with metric $(e^{-2\alpha t})\varphi^T \varphi + \alpha^2 dt^2$ is isometric to $T^{n-1} \times [0, 3\alpha]$ with metric $e^{-2t}I^\varphi + dt^2$, and the isometry is simply $(x, t) \mapsto (x, \alpha t)$.

4.3. Suppose M is a finite volume hyperbolic manifold of dimension n and H a totally geodesic hypersurface with a cusp isometric to $F^{n-2} \times [0, \infty)$ with metric $e^{-2t}S + dt^2$. Then because the radius of injectivity tends to zero only in the cusps, M has a cusp such that the cusp of H is included in the cusp of M . That is, taking $x \in F^{n-2}$, then $t \mapsto (x, t) \in \{x\} \times [0, \infty)$ is a geodesic in H and also in M , and the radius of injectivity, as a function of t , is strictly decreasing, so that there is a t_0 such that for $t > t_0$, the cusp of H is contained in some cusp of M , and cannot leave this cusp because the radius of injectivity outside the cusps is bounded above from zero. Moreover, suppose that the cusp of M is also of the form $F^{n-1} \times [0, \infty)$. Then because the only geodesics that do not leave a cusp are of the form described before (that is, $t \mapsto (x, t) \in \{x\} \times [0, \infty)$) we have that there is a flat F^{n-2} inside the F^{n-1} such that the inclusion of the cusps is simply this inclusion (i.e., $F^{n-2} \hookrightarrow F^{n-1}$) times $[t_0, \infty)$.

4.4. Lemma. *Let M be a compact differentiable n -manifold such that one of the components of the boundary is diffeomorphic to the $(n-1)$ -torus (denote it by T^{n-1}), that is, $T^{n-1} \subset \partial M$. Suppose one of the following holds:*

(i) n is odd;

(ii) n is even and the restriction map $H^{n-2}(M, \partial M \setminus T^{n-1}) \rightarrow H^{n-2}(T^{n-1})$ is not onto.

Then the map $\iota_3 : H^3(DM, DM \setminus T^{n-1}) \rightarrow H^3(DM)$ is not zero. Here DM denotes the double of M .

Remark. We are using \mathbb{Z}_2 coefficients.

Proof. First write DM as $M_0 \cup M_1$, where $M_i = M$, $i = 0, 1$, and we are identifying the boundaries by the identity map. Denote by $p_i : DM \rightarrow M_i$ the map defined by $p_i(x) = x \in M_i$, and $\sigma : DM \rightarrow DM$, defined by $\sigma(x) = p_i(x)$, for $x \in M_{i+1}$, $i \in \mathbb{Z}_2$. Then σ is an involution and p_i is a retraction. Write also $P = T^{n-1} \times [-1, 1]$ for a tubular neighborhood of $T^{n-1} \subset DM$, and assume P σ -symmetric (i.e., $\sigma P = P$). Let $N = DM \setminus \text{int } P$. Then N is contained in $M_0 \cup M_1 \setminus T^{n-1}$; so we consider $M_i \subset N$ and denote by $r_i : M_i \rightarrow N$ the inclusion.

Also, ∂N is homeomorphic to two copies T_0^{n-1} and T_1^{n-1} of T^{n-1} , with $T_i^{n-1} \subset M_i$. (This homeomorphism is the restriction of the tubular neighborhood projection $P \rightarrow T^{n-1}$.) Write $q_i : T^{n-1} \rightarrow T_i^{n-1} \subset \partial N$, $i = 0, 1$, for these homeomorphisms. Also, let $u_i : H^*(M_i) \rightarrow H^*(T^{n-1})$ be the composition of the map induced by the inclusion $T_i^{n-1} \subset M_i$ with q_i^* . Note that $\sigma(T_i^{n-1}) = T_{i+1}^{n-1}$, $\sigma q_i = q_{i+1}$, $i \in \mathbb{Z}_2$ and $\sigma|_{\partial M_0 \setminus T^{n-1}}$ is the identity. Also, write $s_i : T_i^{n-1} \rightarrow N$ for the inclusions. Then $\sigma s_i = s_{i+1}\sigma$, $i \in \mathbb{Z}_2$.

Define $\phi_* : H^*(N) \rightarrow H^*(T^{n-1})$ by $\phi = q_0^* s_0^* + q_1^* s_1^*$. It is not difficult to see that the following diagram is commutative:

$$\begin{array}{ccc} H^k(N) & \cong & H^k(DM \setminus T^{n-1}) \\ \phi \downarrow & & \downarrow \delta \\ H^k(T^{n-1}) & \cong & H^{k+1}(DM, DM \setminus T^{n-1}) \end{array}$$

where the lower isomorphism is excision composed with the canonical isomorphism $H^k(T^{n-1}) \rightarrow H^{k+1}(T^{n-1} \times [0, 1], \partial)$. Let $\epsilon : H^*(M_0, \partial M_0 \setminus T^{n-1}) \rightarrow H^*(N)$ be the composition of the excision map $H^*(M_0, \partial M_0 \setminus T^{n-1}) \rightarrow H^*(N, M_1)$ with the inclusion $H^*(N, M_1) \rightarrow H^*(N)$. Let $\eta : H^*(M_0, \partial M_0 \setminus T^{n-1}) \rightarrow H^*(M_0)$ be the inclusion. We have the commutative diagram

$$\begin{array}{ccccc} H^*(M_0, \partial M_0 \setminus T^{n-1}) & \xrightarrow{\epsilon} & H^*(N) & \xrightarrow{q_0^* s_0^*} & H^*(T^{n-1}) \\ & \searrow \eta & r_0^* \downarrow & u_0 \nearrow & \\ & & H^*(M_0) & & \end{array}$$

Define $\rho = q_0^* s_0^* \epsilon$. Note that this ρ is the map that appears in (ii) of the statement of the lemma.

Consider now the following exact sequence:

$$\rightarrow H^{k-1}(DM) \rightarrow H^{k-1}(DM \setminus T^{n-1}) \xrightarrow{\delta} H^k(DM, DM \setminus T^{n-1}) \xrightarrow{\iota_k} H^k(DM) \rightarrow .$$

Claim. $\iota_k = 0$ implies ρ_{k-1} onto.

Proof of the claim. $\iota_k = 0$ implies δ onto, which in turn implies ϕ_{k-1} onto. Hence, given $c \in H^{k-1}(T^{n-1})$, there is $b \in H^{k-1}(N)$, with $\phi_{k-1} b = c$ (we just write $\phi b = c$). Let $d = b - \sigma^* b \in H^{k-1}(N)$, and $a = r_0^* d = d|_{M_0}$. Note that $a|_{\partial M_0 \setminus T^{n-1}} = d|_{M_0}|_{\partial M_0 \setminus T^{n-1}} = d|_{\partial M_0 \setminus T^{n-1}} = (b - \sigma^* b)|_{\partial M_0 \setminus T^{n-1}} = b|_{\partial M_0 \setminus T^{n-1}} - \sigma^*(b|_{\partial M_0 \setminus T^{n-1}}) = 0$ because $\sigma|_{\partial M_0 \setminus T^{n-1}} = Id$. Hence there is $e \in H^{k-1}(M_0, \partial M_0 \setminus T^{n-1})$ with $a = \eta e$.

Then, we get $\rho e = q_0^* s_0^* \epsilon e = u_0 \eta e = u_0 a = u_0 r_0^* d = q_0^* s_0^* d = q_0^* s_0^* (b - \sigma^* b) = q_0^* s_0^* b - q_0^* s_0^* \sigma^* b = q_0^* s_0^* b - q_0^* \sigma^* s_1^* b = q_0^* s_0^* b - q_1^* s_1^* b = q_0^* s_0^* b + q_1^* s_1^* b = \phi b = c$. (Recall that we are using \mathbb{Z}_2 coefficients.) This proves the claim.

Consequently, if $\iota_3 = 0$, then ρ_2 is onto, so that ρ_4 is also onto (this is because ρ is a ring homomorphism and also because $H^2(T^{n-1}) \otimes H^2(T^{n-1}) \rightarrow H^4(T^{n-1})$, given by the cup product, is an epimorphism). Repeating this process we obtain that ρ_{2k} is onto. Now if n is odd, this means that ρ_{n-1} is onto. This is a contradiction, because $\rho_{n-1} = 0$ and $H^{n-1}(T^{n-1})$ is not zero. If n is even, then ρ_{n-2} is onto, which contradicts (ii) of the statement of the lemma. This completes the proof.

4.5. Suppose that we have a totally geodesic hypersurface H with a cusp of the form $T^{n-2} \times [0, \infty)$ contained in a cusp C of the manifold M , also of the form $T^{n-1} \times [0, \infty)$. Suppose also that this is the only cusp of H contained in the cusp C of M (this last statement makes sense; see 4.3). Then (ii) of lemma (4.4) holds for M . To prove this, let $x \in H^{n-2}(T^{n-1})$ be the cohomology class dual to a loop $\gamma \subset T^{n-1} \subset C$ that intersects $T^{n-2} \subset T^{n-1}$ transversally in one point (see 4.8). Let also $y \in H^{n-2}(\partial M, \partial M \setminus T^{n-1})$ be the image of x under the excision isomorphism $H^{n-2}(T^{n-1}) \cong H^{n-2}(\partial M, \partial M \setminus T^{n-1})$. Then y is the restriction of a class in $H^{n-2}(M, \partial M \setminus T^{n-1})$ if and only if $\delta'(y) = 0$, where $\delta' : H^{n-2}(\partial M, \partial M \setminus T^{n-1}) \rightarrow H^{n-1}(M, \partial M)$ denotes the connecting homomorphism in the exact sequence of the

triple $(M, \partial M, \partial M \setminus T^{n-1})$. Since γ intersects H transversally in one point, $\delta'(y)$ evaluates to 1 on the fundamental class of the hypersurface H . Consequently, $\delta'(y) \neq 0$, and x is not the restriction of a class in $H^{n-2}(\partial M, \partial M \setminus T^{n-1})$.

We will need some results about involutions on a flat torus.

4.6. Lemma. *Let $\iota \neq 1$ be an isometric involution on a flat torus T^n . Suppose that there is a (connected) totally geodesic flat $(n-1)$ -torus N contained in the fixed point set $\text{fix}(\iota)$ of ι . Then, either*

- (i) $N = \text{fix}(\iota)$, or
- (ii) *there is an isometry $\eta : T^n \rightarrow \mathbb{S}_b^1 \times N$, for some $b > 0$, such that $\eta\iota\eta^{-1}(z, x) = (\bar{z}, x)$.*

Here $\mathbb{S}_b^1 = \{z \in \mathbb{C} : |z| = b\}$ and the bar denotes conjugation.

Proof. Let $L \subset \mathbb{R}^n$ be a lattice generated by $v_1, \dots, v_n \in \mathbb{R}^n$ such that $\mathbb{R}^n/L \cong_{ISO} T^n$ (\mathbb{R}^n with its standard metric) and such that $\pi(0) \in N$, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n/L \cong_{ISO} T^n$ is the covering projection.

Let $\tilde{\iota} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a lifting of ι , such that $\tilde{\iota}(0) = 0$. Let H be the component of $\pi^{-1}(N)$ that contains 0. Then H is an $(n-1)$ -linear subspace of \mathbb{R}^n and $\tilde{\iota}|_H = Id$. Hence $\tilde{\iota}$ is a Euclidean reflection, i.e., $\tilde{\iota}(x) = x - 2\langle x, v \rangle v$, where v is orthonormal to H .

We can suppose (after a rotation) that $H = \{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$ or, equivalently, that $v = e_1 = (1, 0, \dots, 0)$. Let $a = \min\{t > 0 : te_1 + H \subset L + H\}$. Then $a > 0$ and $ae_1 + H \subset L + H$.

Let $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be the projection to the first coordinate $x \mapsto \langle e_1, x \rangle$.

Claim 1. a generates the subgroup $p(L)$ of \mathbb{R} .

To see this, note that $t \in p(L)$ iff $te_1 + H \subset L + H$, and because $p(L)$ is a discrete subgroup of \mathbb{R} , then it is generated by $\min\{t > 0 : te_1 + H \subset L + H\} = a$.

Claim 2. $2ae_1 \in L$.

Because $\iota|_N = Id$ and $\pi(ae_1) \in \pi(L + H) = \pi(H) = N$, we get $-ae_1 = \tilde{\iota}(ae_1) = ae_1 + u$, for some $u \in L$. Hence $2ae_1 = -u \in L$. This proves claim 2.

By claims 1 and 2, $L \cap \mathbb{R}e_1$ is generated either by $2ae_1$ or ae_1 .

Claim 3. $\pi^{-1}(\text{fix}(\iota)) = \bigcup \{se_1 + H : 2se_1 \in L\}$.

Given $x \in \mathbb{R}^n$, write $x = se_1 + y$, $y \in \{0\} \times \mathbb{R}^{n-1}$. Then $x \in \pi^{-1}(\text{fix}(\iota))$ iff $2se_1 = x - \tilde{\iota}(x) \in L$. The claim follows.

To finish the proof we have two cases.

First case. $ae_1 \notin L$.

In this case, $2ae_1$ generates $L \cap \mathbb{R}e_1$. Thus $2se_1 \in L$ iff $s = ma$, for some $m \in \mathbb{Z}$. Hence $\pi^{-1}(\text{fix}(\iota)) = \bigcup_{m \in \mathbb{Z}} mae_1 + H$ and we get $\text{fix}(\iota) = \pi(\pi^{-1}(\text{fix}(\iota))) = \pi(\bigcup_{m \in \mathbb{Z}} mae_1 + H) \subset \pi(L + H) = \pi(H) = N$. Consequently, $N = \text{fix}(\iota)$, and this is (i) of the statement of the theorem.

Second case. $ae_1 \in L$.

In this case, ae_1 generates $L \cap \mathbb{R}e_1$. Thus $2se_1 \in L$ iff $s = \frac{1}{2}ma$, for some $m \in \mathbb{Z}$. Hence $\pi^{-1}(\text{fix}(\iota)) = \bigcup_{m \in \mathbb{Z}} \frac{1}{2}ma + H$ and we get $\text{fix}(\iota) = \pi(\pi^{-1}(\text{fix}(\iota))) = \pi(\bigcup_{m \in \mathbb{Z}} \frac{1}{2}ma + H) = \pi(H) \cup \pi(\frac{1}{2}ae_1 + H) = N + N'$, where $N' = \pi(\frac{1}{2}ae_1 + H)$ is disjoint from N .

Now, because a is minimum, each v_i can be written as $v_i = m_i a e_1 + y_i$, $y_i \in H$, $m_i \in \mathbb{Z}$. But $a e_1 \in L$ implies $y_i \in L$. Hence we have that $L = \mathbb{Z} a e_1 \oplus (L \cap H)$. Then $\pi_1 T^n \cong \pi_1 N \oplus \mathbb{Z}$, where \mathbb{Z} is generated by the loop $\pi(\{t e_1 : 0 \leq t \leq 1\})$, orthogonal to N . The lemma follows.

Remarks. **1.** If $a e_1 \notin L$, that is, if $N = \text{fix}(\iota)$, we say that N is a *double* $(n-1)$ -subtorus of T^{n-1} , or, that ι is a *double* isometric involution. If $a e_1 \in L$, that is, if (ii) of the statement of the lemma above holds, we say that N is a *simple* $(n-1)$ -subtorus of T^{n-1} , or, that ι is a *simple* isometric involution.

2. Note that if ι is simple, then $\pi_1 T^n \cong \mathbb{Z} e \oplus L_1$, where $\iota_*|_{L_1} = \text{Id}$ and $\iota_* e = -e$. Here $\iota_* : \pi_1 T^n \rightarrow \pi_1 T^n$ is the induced homomorphism. The next corollary proves the converse.

4.7. Corollary. *Let $\iota : T^n \rightarrow T^n$ be an isometric involution such that $\pi_1 T^n \cong \mathbb{Z} e \oplus L_1$, with $\iota_*|_{L_1} = \text{Id}$ and $\iota_* e = -e$. Then ι is simple.*

Proof. We use the notation of the proof of lemma 4.6. Hence, we identify $\pi_1 T^n$ with the lattice L and L_1 with $L \cap H$.

If $a e_1 \in L$, we are done, by definition of simple. Suppose $a e_1 \notin L$. Then, we should have $L = \mathbb{Z}(2a e_1) \oplus (L \cap H)$. But $a e_1 + H \subset L + H = [\mathbb{Z}(2a e_1) + (L \cap H)] + H = \mathbb{Z}(2a e_1) + H$. Hence $a e_1 = 2m a e_1 + h$, for some $h \in H$, $m \in \mathbb{Z}$. This is a contradiction because $\langle e_1, h \rangle = 0$ and $a \neq 0$.

4.8. Lemma. *Let T_i , $i = 1, 2, 3$, be three mutually orthogonal flat $(n-1)$ -subtori of a flat n -torus T^n , such that, for each i , T^n is isometric to $T_i \times \mathbb{S}_{b_i}^1$, for some $b_i > 0$. Then the flat codimension-three totally geodesic submanifold $N = \bigcap_{i=1}^3 T_i$ is connected.*

Proof. Let $\pi : \mathbb{R}^n \rightarrow T^n$ be the covering projection, where we consider \mathbb{R}^n with its canonical metric. We can assume, after a rotation, that $\pi(P_i) = T_i$, $i = 1, 2, 3$, where $P_i = \{x_i = 0\}$. Let L be a lattice such that $\mathbb{R}^n/L \cong_{ISO} T^n$. Hence $L = \mathbb{Z} e_1 \oplus \mathbb{Z} e_2 \oplus \mathbb{Z} e_3 \oplus (L \cap P_1 \cap P_2 \cap P_3)$ and $N = \pi(\{x_1 = x_2 = x_3 = 0\})$. The result follows.

4.9. Lemma. *Let N^k be a totally geodesic compact k -submanifold of a flat n -torus T^n . Then there is a totally geodesic compact $(n-k)$ -submanifold intersecting N transversally in one point.*

Proof. Let $L \subset \mathbb{R}^n$ be a lattice generated by $v_1, \dots, v_n \in \mathbb{R}^n$ such that $\mathbb{R}^n/L \cong_{ISO} T^n$ (\mathbb{R}^n with its standard metric) and such that $\pi(0) \in N$, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n/L \cong_{ISO} T^n$ is the covering projection.

Let $H = \pi^{-1}(N)$. Then H is a k -linear subspace of \mathbb{R}^n and, after a rotation, we can suppose $H = \{0\} \times \mathbb{R}^k \subset \mathbb{R}^{n-k} \times \mathbb{R}^k = \mathbb{R}^n$. Let $p : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ be the projection. Then $p(L)$ is an abelian discrete cocompact subgroup of \mathbb{R}^{n-k} , that is, it is a lattice generated by some a_1, \dots, a_{n-k} . Let $v_i \in L$ be such that $p(v_i) = a_i$, $i = 1, \dots, n-k$.

Let $(b, y) \in L$, $b \in \mathbb{R}^{n-k}$, $y \in \mathbb{R}^k$. We write $v_i = (a_i, y_i)$, for some $y_i \in \mathbb{R}^k$. Because the a_i 's generate $p(L)$, $b = \sum m_i a_i$, $m_i \in \mathbb{Z}$. Hence,

$$(b, y) = \sum m_i v_i + [(b, y) - \sum m_i v_i] = \sum m_i v_i + (0, y - \sum m_i y_i),$$

which implies that $(0, y - \sum m_i y_i) \in L \cap \mathbb{R}^k$. This proves that $L = (L \cap \mathbb{R}^k) \oplus L_1$, where $L_1 \subset L$ is the lattice of \mathbb{R}^{n-k} generated by v_1, \dots, v_{n-k} . Let J be the $(n-k)$ -subspace generated by the v_i 's. Then it is easy to see that the compact totally geodesic $(n-k)$ -submanifold $P = \pi(J)$ satisfies the conditions of the lemma.

Finally, we will need the following result about geodesic spaces (see [6] for definitions).

4.10. Lemma. *Let X be a compact non-positively curved geodesic space. Then there exists an abelian subgroup of rank k in $\pi_1(X, *)$ if and only if there is a flat k -torus immersed isometrically and totally geodesically in X . If X is a PL space with a piecewise flat metric, the immersion is also PL.*

Proof. The first part is the well-known flat torus theorem for geodesic spaces (see [6]). To complete the proof of the theorem, assume that X is piecewise flat, and we have to prove that the immersion of the torus T^k in X is PL, but this follows from the next trivial lemma.

4.11. Lemma. *Let $g : A \rightarrow K$ be an isometric and totally geodesic embedding of piecewise flat finite simplicial complexes. Then g is PL.*

Proof. Totally geodesic isometries send geodesics to geodesics linearly (in fact, by translations). Hence the lemma follows by definition (use the definition given in [16]).

5. PROOF OF THEOREM 3

Theorem 3. *Let M be a non-compact finite volume hyperbolic n -manifold, $n > 5$, with a large cusp, C_M , diffeomorphic to $T^{n-1} \times (0, \infty)$. Assume either*

(1) *n is odd, or*

(2) *n is even and there is a totally geodesic hypersurface $N \subset M$, with exactly one cusp contained in the cusp C_M of M .*

Then the double DM of M admits (at least) two different (smoothable) PL structures admitting Riemannian metrics with non-positive curvature.

Proof. Consider $T^{n-1} \times (0, 3)$ with canonical flat metric $I + dt^2$ (see 2.11). By the triangulation lemma 3.2 for every element $c = c_\tau \in H^3(T^{n-1} \times (0, 3), T^{n-1} \times (0, 3) \setminus T^{n-1} \times \{2\}) \cong H^2(T^{n-1})$, there is a PL homeomorphism χ_τ such that if we glue $T^{n-1} \times (0, 2]$ with $T^{n-1} \times [2, 3)$ by identifying $(x, 2) \in T^{n-1} \times (0, 2]$ with $(\chi_\tau(x), 2) \in T^{n-1} \times [2, 3)$ we obtain a triangulation non-concordant (modulo the complement of $T^{n-1} \times \{2\}$) with the canonical one (that is, the one induced by the flat structure). We know that all these triangulations are smoothable (see [17] p. 227) so that we suppose all the χ_τ differentiable (see 3.3).

Because the set of metrics is convex, for each $\tau \in H^3(T^{n-1} \times (0, 3), T^{n-1} \times (0, 3) \setminus T^{n-1} \times \{2\})$ we can choose a metric B_τ on $T^{n-1} \times [1, 2]$ satisfying (2.1) and also the following:

- (a) $B_\tau|_{T^{n-1} \times \{1\}} = I|_{T^{n-1} \times \{1\}}$,
- (b) $B_\tau|_{T^{n-1} \times \{2\}} = \chi_\tau^* I|_{T^{n-1} \times \{2\}}$, where $\chi = \chi_\tau$,
- (c) $B_\tau|_{T^{n-1} \times \{t\}}$ is constant near $t = 1, 2$.

Define now a metric A_τ on $(T^{n-1} \times (0, 3), \tau)$ by

$$A_\tau = \begin{cases} B_\tau & t \in [1, 2], \\ I & t \in (0, 1] \cup [2, 3). \end{cases}$$

We apply now lemma (2.10) to each A_τ , $\epsilon < 1$, and $f(\alpha, t) = e^{4\alpha-t}$ (note that this f satisfies (2.3) for $t \in [0, 3]$ with $-\ell_1 = \ell_2 = 1$). Let L be the maximum over all elements $c_\tau \in H^3(T^{n-1} \times (0, 3), T^{n-1} \times (0, 3) \setminus T^{n-1} \times \{2\})$ of the constants we get from lemma (2.10). This means that the metrics $(A_\tau)_\alpha^\varphi$ are non-positively curved for $\alpha > L$ and that φ is distance non-decreasing (this is because $-\ell_1 = \ell_2 = 1$) and $\epsilon < 1$. By taking ϵ small, we can have curvature as close to -1 as we want.

5.1. Now define $c_n = e^{4(L+1)}$.

We now try to fit $T^{n-1} \times (0, 3)$ with these non-positively curved exotic triangulations on the double DM of M . Recall that we make no distinction between the cusp and $T^n \times [0, \infty)$.

Let M be as in the statement of the theorem and also let one of the cusps be isometric to $T^{n-1} \times [0, \infty)$ with metric $e^{-2t}I\varphi + dt^2$. Assume that the cusp is large. That means that $(\frac{1}{c_n})\varphi = (\frac{1}{e^{4(L+1)}})\varphi$ is distance non-decreasing. This implies that $(A_\tau)_{L+1}^{(\frac{1}{e^{4(L+1)}})\varphi}$ is non-positively curved.

Consider the inclusion $T^{n-1} = T^{n-1} \times \{2\alpha\} \subset T^{n-1} \times [\alpha, 2\alpha] \subset M \subset DM$ (to be able to double M , we are cutting the cusp far away from $T^{n-1} \times [\alpha, 2\alpha]$, that is, choose $b > 2\alpha$ in section 1). Then DM_{χ_τ} (obtained by cutting along $T^{n-1} \times \{2\alpha\}$ and gluing back with χ_τ) admits a non-positively curved metric: define it to be the old one (the hyperbolic metric) outside $T^{n-1} \times [\alpha, 2\alpha]$ and inside $T^{n-1} \times [\alpha, 2\alpha]$ to be the pullback of $(A_\tau)_{L+1}^{(\frac{1}{e^{4(L+1)}})\varphi}$ by the diffeomorphism $T^{n-1} \times (0, 3\alpha) \rightarrow T^{n-1} \times (0, 3)$, $(x, t) \mapsto (x, \frac{t}{\alpha})$. Note that this metric is well defined because $(A_\tau)_{L+1}^{(\frac{1}{e^{4(L+1)}})\varphi}$ coincides with $I_{L+1}^{(\frac{1}{e^{4(L+1)}})\varphi} + \alpha^2 dt^2$ for $t \in (0, 1] \cup [2, 3)$ and $T^{n-1} \times (0, 3)$ with this last metric is isometric to $T^{n-1} \times (0, 3\alpha)$ with metric $e^{-2t}I\varphi + dt^2$ (see 4.2).

Thus, doubling M , for each $\tau \in H^3(T^{n-1} \times (0, 3), T^{n-1} \times (0, 3) \setminus T^{n-1} \times \{2\})$, we obtain in this way non-positively curved triangulations non-concordant to the canonical one modulo the complement of $T^{n-1} \times \{2\alpha\}$.

Now, if n is odd, lemma 4.4(i) tells us that at least one of these τ 's is non-concordant to the canonical triangulation. If n is even, then condition (2) of the theorem together with remark 4.5 imply again that at least one of these τ 's is non-concordant to the canonical triangulation, and corollary 9.6 implies that this non-concordant PL structure is, in fact, not equivalent to the canonical one. This completes the proof of theorem 3.

6. PROOF OF THEOREM 4

Here we prove theorem 4.

Theorem 4. *Let M, N be two non-compact finite volume real hyperbolic manifolds such that*

- (1) *N is a totally geodesic submanifold of M ,*
- (2) *$\dim M = n \geq 5$, $\dim N = n - 3$,*
- (3) *the normal bundle of N is trivial,*
- (4) *M has a cusp C_M diffeomorphic to $T^{n-1} \times (0, \infty)$,*
- (5) *N has exactly one cusp contained in C_M .*

Then the double DM of M has a differentiable structure \mathcal{E} , such that

- (i) *(DM, \mathcal{E}) is not PL equivalent to DM , and*

(ii) (DM, \mathcal{E}) does not admit a Riemannian metric with non-positive curvature.

In fact, DM does not even admit a piecewise flat metric with non-positive curvature.

Proof. Note first that $DN \subset DM$ also has trivial normal bundle and we denote it by $\mathbb{D}^3 \times DN$. Note also that the cross section of C_M is diffeomorphic to T^{n-1} , and we just write $T^{n-1} \subset C_M \subset M \subset DM$, and because N is totally geodesic in M , $DN \cap T^{n-1} = N \cap T^{n-1}$ is a flat $(n-4)$ -torus that we denote by T^{n-4} .

Recall that the set of concordance classes of PL structures on DM is in one-to-one correspondence with $H^3(DM, \mathbb{Z}_2)$ and assume that the PL structure induced by the hyperbolic structure on M corresponds to $0 \in H^3(DM, \mathbb{Z}_2)$. Recall also that $H^3(DM, \mathbb{Z}_2) \cong [DM, TOP/PL]$, the set of homotopy classes of maps from DM to TOP/PL .

Consider the map

$$DM \rightarrow DM/[DM \setminus (\mathbb{D}^3 \times DN)] \cong \mathbb{D}^3 \times DN / \partial(\mathbb{D}^3 \times DN) \rightarrow \mathbb{D}^3 / \partial \mathbb{D}^3 \xrightarrow{\alpha} TOP/PL$$

where the first maps are collapsing maps and $\alpha : \mathbb{D}^3 / \partial \mathbb{D}^3 \rightarrow TOP/PL$ is the generator of $\pi_3(TOP/PL) \cong \mathbb{Z}_2$. Call this map $\theta : DM \rightarrow TOP/PL$. Then θ corresponds to the cohomology class $c \in H^3(DM, \mathbb{Z}_2)$ dual to the homology class determined by DN , and this determines a PL structure Σ on DM . Σ is not concordant to the canonical PL structure on DM because the homology class determined by DN is not zero (DN intersects transversally a three torus $T^3 \subset T^{n-1}$ in one point; see 4.8). Because $\pi_3(TOP/PL) \cong \pi_3(TOP/O)$, Σ is smoothable, so that it is induced by a differentiable structure \mathcal{E} on DM . Also, because of corollary 9.6, Σ (or \mathcal{E}) is not PL equivalent to the canonical PL structure.

Consider now the covering corresponding to $\pi_1(T^{n-1}) \subset \pi_1(DM)$. This covering is diffeomorphic to $T^{n-1} \times \mathbb{R}$. Denote by $p : T^{n-1} \times \mathbb{R} \rightarrow DM$ the covering projection. Because of lemma 3.1, $p^*\Sigma$ corresponds to p^*c .

Claim. $p^*c = \text{dual}(T^{n-4} \times \mathbb{R}) \in H^3(T^{n-1} \times \mathbb{R}, \mathbb{Z}_2)$.

Proof of the claim. $T^{n-1} \subset M \subset DM$ has a normal neighborhood $U \cong T^{n-1} \times [0, 1]$, and note that $U \cap DN \cong T^{n-4} \times [0, 1]$. This normal neighborhood lifts to a normal neighborhood U' , $U' \subset T^{n-1} \times \mathbb{R}$, such that $p|_{U'} : U' \rightarrow U$ is a homeomorphism and it sends $p^{-1}(DN) \cap U'$ homeomorphically onto $DN \cap U \cong T^{n-4} \times [0, 1]$. Then $p^{-1}(DN) \cap U' = (T^{n-4} \times \mathbb{R}) \cap U'$, but the restriction map $H^3(T^{n-1} \times \mathbb{R}) \rightarrow H^3(U')$ induced by the inclusion is an isomorphism. The claim follows.

Let T^{n-1} be the PL exotic (differentiable) $(n-1)$ -torus whose PL structure corresponds to the cohomology class dual to $T^{n-4} \subset T^{n-1}$. Then the claim implies

$$(T^{n-1} \times \mathbb{R}, p^*\mathcal{E}) \cong_{PL} T^{n-1} \times \mathbb{R},$$

and by choosing the right differentiable structure on T^{n-1} , we even have

$$(T^{n-1} \times \mathbb{R}, p^*\mathcal{E}) \cong_{DIFF} T^{n-1} \times \mathbb{R}.$$

Suppose now that (DM, \mathcal{E}) admits a non-positively curved Riemannian metric. Then, by [13], there is an isometric (differentiable) embedding $j : T^{n-1} \rightarrow (DM, \mathcal{E})$ (T^{n-1} here has its canonical differentiable structure) such that $j_*(\pi_1(T^{n-1})) = \pi_1(T^{n-1}) \subset \pi_1(DM)$. Then j lifts to an embedding $j' : T^{n-1} \rightarrow (T^{n-1} \times \mathbb{R}, p^*\mathcal{E})$.

$\mathbb{R}, p^*\mathcal{E}) \cong_{DIFF} \mathcal{T}^{n-1} \times \mathbb{R}$. But this implies that T^{n-1} and \mathcal{T}^{n-1} are differentiably h -cobordant. Hence $T^{n-1} \cong_{DIFF} \mathcal{T}^{n-1}$ (because $Wh(\pi_1(T^{n-1})) = 0$), a contradiction.

To see that (DM, \mathcal{E}) does not even admit a piecewise flat non-positively curved metric, use lemma 4.9 and proceed in the same way. That is, suppose now that (DM, \mathcal{E}) admits a piecewise flat non-positively curved metric. Then, by lemma 4.9, there is an isometric PL embedding $j : T^{n-1} \rightarrow (DM, \mathcal{E})$ (T^{n-1} here has its canonical PL structure) such that $j_*(\pi_1(T^{n-1})) = \pi_1(T^{n-1}) \subset \pi_1(DM)$. Then j lifts to an embedding $j' : T^{n-1} \rightarrow (T^{n-1} \times \mathbb{R}, p^*\mathcal{E}) \cong_{DIFF} \mathcal{T}^{n-1} \times \mathbb{R}$. But this implies that T^{n-1} and \mathcal{T}^{n-1} are PL h -cobordant. Hence $T^{n-1} \cong_{PL} \mathcal{T}^{n-1}$ (because $Wh(\pi_1(T^{n-1})) = 0$), a contradiction. This proves theorem 4.

7. THE MANIFOLDS $M(\mathcal{Q}, a)$ AND PROOF OF THEOREM 5

Let \mathcal{Q} be a quadratic form of signature $(n, 1)$ with rational coefficients, $O(\mathcal{Q})$, the group of $n + 1$ square matrices with real entries that preserve \mathcal{Q} and $O(\mathcal{Q})_{\mathbb{Z}} = O(\mathcal{Q}) \cap GL(n+1, \mathbb{Z})$. Let $C \in GL(n+1, \mathbb{R})$ be such that $C^T \mathcal{Q} C = \text{diag}\{1, \dots, 1, -1\}$. Hence $C^{-1}(O(\mathcal{Q})_{\mathbb{Z}})C \subset C^{-1}O(\mathcal{Q})C \subset O(n, 1; \mathbb{R})$. $O(\mathcal{Q})/O(\mathcal{Q})_{\mathbb{Z}}$ has finite volume (see [5]). In general, $O(\mathcal{Q})_{\mathbb{Z}}$ is not torsion free, but it has finite index subgroups that are torsion free. These subgroups are the principal congruence subgroups defined in the following way. Let \mathcal{R} be a commutative ring with unity, \mathcal{I} an ideal of \mathcal{R} such that \mathcal{R}/\mathcal{I} is finite and Γ a group of square n by n matrices with entries in \mathcal{R} . Then the congruence subgroup $\Gamma_{\mathcal{I}}$ is the subgroup of Γ of all the matrices congruent to the identity modulo \mathcal{I} . Then $\Gamma_{\mathcal{I}}$ is of finite index. In our case, if we take \mathbb{Z} (or any ideal of it) as being our ring \mathcal{R} and Γ as $O(\mathcal{Q})_{\mathbb{Z}}$, it is known that (see [4]) for all but finite prime ideals \mathcal{P} , $\Gamma_{\mathcal{P}}$ is torsion free, so that $C^{-1}\Gamma_{\mathcal{P}}C \backslash O(n+1, 1; \mathbb{R})/O(n, \mathbb{R})$ is a complete finite volume hyperbolic manifold. Then for the integers a , such that $(O(\mathcal{Q})_{\mathbb{Z}})_a$ (i.e., take congruence mod a) is torsion free, we receive a finite volume hyperbolic manifold and we denote it by $M(\mathcal{Q}, a)$.

We say that $x \in \mathbb{R}^{n+1}$ is an isotropic vector of the quadratic form \mathcal{Q} if $x^T \mathcal{Q} x = 0$. Isotropic lines (i.e., lines determined by isotropic vectors) correspond to points in the sphere at infinity of the hyperbolic space \mathbb{H}^n . (This is easy to see if we use the hyperboloid model of \mathbb{H}^n .) The following lemma tells us that to a rational isotropic vector of the rational quadratic form \mathcal{Q} of signature $(n, 1)$ corresponds a cusp.

7.1. Lemma. *Let a be an integer. If $x \in \mathbb{Q}^{n+1}$ is such that $x^T \mathcal{Q} x = 0$, then $\{g \in (O(\mathcal{Q})_{\mathbb{Z}})_a : gx = x, g \text{ parabolic}\}$ has more than one element. Conversely, if x is isotropic and $\{g \in (O(\mathcal{Q})_{\mathbb{Z}})_a : gx = x, g \text{ parabolic}\}$ is not the trivial group, then we can choose $x \in \mathbb{Q}^{n+1}$.*

Remarks.

7.1.1. The subgroup $\{g \in (O(\mathcal{Q})_{\mathbb{Z}})_a : gx = x, g \text{ parabolic}\}$ is “the fundamental group of the cusp corresponding to x ”.

7.1.2. If we delete from $O(\mathcal{Q})_{\mathbb{Z}}$ the elements of finite order (for instance by taking a congruence subgroup), the condition “being parabolic” would be redundant (if g is hyperbolic and fixes a point at infinity corresponding to the isotropic vector x , then x is an eigenvector of g , but $gx \neq x$).

Proof of Lemma 7.1. The first part follows from the proof of theorem 5, below (we only use the second part of the lemma in the proof of this theorem). In fact,

there we give explicit formulas to find a whole subgroup of $\{g \in (O(\mathcal{Q})_{\mathbb{Z}})_a : gx = x, g \text{ parabolic}\}$ isomorphic to \mathbb{Z}^{n-1} .

Suppose then that $\{g \in (O(\mathcal{Q})_{\mathbb{Z}})_a : gx = x, g \text{ parabolic}\}$ is not the trivial group, and $x^T \mathcal{Q}x = 0$. Take $g \in \{g \in (O(\mathcal{Q})_{\mathbb{Z}})_a : gx = x, g \text{ parabolic}\}$, g different from the identity I . Let $V \subset \mathbb{R}^{n+1}$ be the kernel of $g - I$. Because g is parabolic it acts freely on the hyperboloid $x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1$, which means that $V \cap \{y \in \mathbb{R}^{n+1}, y^T \mathcal{Q}y < 0\}$ is empty. Also, because x is the only isotropic vector fixed by g (g fixes only one point at infinity) we have that $y^T \mathcal{Q}y > 0$ for $y \in V \setminus \mathbb{R}x$. This implies that $\mathcal{Q}|_V$ is degenerate and $\mathbb{R}x$ is the only line in V such that $y^T \mathcal{Q}x = 0$ for all $y \in V$.

On the other hand, because V is the kernel of a matrix with integer coefficients, there is a basis $\{z_1, \dots, z_k\}$ of V where $z_i \in \mathbb{Q}^{n+1}$. If x is a multiple of one z_i , we are done. If not, write $x = \sum \lambda_i z_i$ and the λ_i are uniquely determined (modulo multiplication by a constant) by the set of linear equations with rational coefficients: $z_i^T \mathcal{Q}x = 0$, $i = 1, \dots, k$. Hence we can assume that the λ_i 's are rational and this completes the proof of the lemma.

We now prove theorem 5.

Theorem 5. *If $M(\mathcal{Q}, a)$ is as before, then it admits finite covers $M(\mathcal{Q}, ba)$ with one cusp diffeomorphic to $T^{n-1} \times (0, \infty)$.*

Proof. Recall that, in general, if Σ is a discrete torsion free subgroup of $O(n, 1; \mathbb{R})$ such that $\Sigma \backslash O(n, 1; \mathbb{R}) / O(n-1, \mathbb{R})$ has finite volume and is non-compact, then the cusps are of the form $F \times [0, \infty)$ with metric $e^{-2t}S + dt^2$, for some flat metric S on the compact manifold F and $F \cong \mathbb{R}^{n-1} / \Lambda$ (see [2] sec. D3), where $\Lambda \subset \Sigma$ is the subgroup of elements that fix one point in the sphere at infinity. Because the points at the sphere at infinity correspond to isotropic lines, we can write $\Lambda = \{g \in \Sigma : gx = x\}$ for some isotropic vector x (see remarks 7.1.1 and 7.1.2).

In our case, because of the lemma above, we can suppose that $M(\mathcal{Q}, a)$ has a cusp that corresponds to $x \in \mathbb{Q}^{n+1}$. If we define $G = \{g \in O(\mathcal{Q}) : gx = x\}$, to prove the theorem it is enough to prove the following.

7.2. Lemma. $G_{\mathbb{Z}} = G \cap GL(n+1, \mathbb{Z})$ has a congruence subgroup $(G_{\mathbb{Z}})_a$ isomorphic to \mathbb{Z}^{n-1} .

Proof. In what follows, we write $v.w$ instead of $Q(v, w) = v^T \mathcal{Q}w$, for $v, w \in \mathbb{R}^{n+1}$. Because \mathcal{Q} is rational of signature $(n, 1)$, there is a basis $\mathcal{B} = \{r_1, \dots, r_{n-1}, x_0, e\} \subset \mathbb{Q}^{n+1}$ for \mathbb{R}^{n+1} satisfying

$$\begin{aligned} r_1, \dots, r_{n-1}, e &\in \mathbb{Z}^{n+1}, \\ x_0 &= \alpha x, \quad \alpha \in \mathbb{Q}, \\ e.r_i &= r_i.r_j = x_0^2 = x_0.r_i = 0 \quad i, j = 1, \dots, n-1 \quad i \neq j, \\ e^2 &< 0, \\ r_i^2 &> 0, \\ x_0.e &= -1. \end{aligned}$$

Note that if $g \in G$ and $ge = a_1 r_1 + \dots + a_{n-1} r_{n-1} + a_n x_0 + a_{n+1} e$, then $a_{n+1} = 1$ because $-1 = x_0.e = gx_0.ge = x_0.ge = a_{n+1}x_0.e = -a_{n+1}$.

We have four steps.

Step 1. Here we express G as a semidirect product of two subgroups. For each $a = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$, define the linear isomorphism $t_a : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$:

$$(a) \ t_a(r_i) = a_i r_i^2 x_0 + r_i, \ i = 1, \dots, n-1,$$

$$(b) \ t_a(x_0) = x_0,$$

$$(c) \ t_a(e) = a_1 r_1 + \dots + a_{n-1} r_{n-1} + a_n x_0 + e, \text{ where } a_n = \frac{1}{2} \sum_{i=1}^{n-1} a_i^2 r_i^2.$$

Then t_a written in the basis $\mathcal{B} = \{r_1, \dots, r_{n-1}, x_0, e\}$ is given by the matrix $(t_a)_{\mathcal{B}} = I + A_a$, where I is the identity matrix and

$$A_a = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} \\ a_1 r_1^2 & \cdot & \cdot & a_{n-1} r_{n-1}^2 & 0 & \cdot & a_n \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

Note that this linear transformation preserves \mathcal{Q} since $t_a(\mathcal{B})$ has the same products as \mathcal{B} (for instance, $t_a(x_0).t_a(e) = x_0.e = -1$). Then if B is the matrix whose columns are the canonical coordinates of $r_1, \dots, r_{n-1}, x_0, e$ we have that $B(t_a)_{\mathcal{B}} B^{-1} \in G$ and we denote this matrix also by t_a . It is easy to verify that $t_{a+b} = t_a.t_b$ so that the map $a \mapsto t_a$ is a homomorphism. Define now

$$T = \{t_a : a \in \mathbb{R}^{n-1}\},$$

$$R = \{g \in G : ge = e\}.$$

It is easy to see that T is an abelian subgroup of G isomorphic (as a Lie group) to \mathbb{R}^{n-1} and $T \cap R = \{I\}$.

Note also that R is isomorphic to $O(n-1, \mathbb{R})$ (because g fixes x_0 and e , which implies that $g(r_i)$ is a linear Q -orthogonal combination of the r_j 's).

Remark that if $g \in G$, and

$$\begin{pmatrix} a_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \\ a_n \\ 1 \end{pmatrix} = (ge)_{\mathcal{B}}$$

(i.e., $ge = a_1 r_1 + \dots + a_{n-1} r_{n-1} + a_n x_0 + e$), then $t_a e = ge$ where $a = (a_1, \dots, a_{n-1})$, so that $t_a^{-1} g \in R$. This implies that $G = TR = \{tg : t \in T, g \in R\}$.

Step 2. Here we show that for all but finite primes p , $(T_{\mathbb{Z}} R_{\mathbb{Z}})_p = (T_{\mathbb{Z}})_p$.

To see this, note that because $O(Q)_{\mathbb{Z}}$ is discrete, we have that $R_{\mathbb{Z}}$ is also discrete. Then, since R is compact, we get that $R_{\mathbb{Z}}$ is finite, i.e., $R_{\mathbb{Z}} = \{I = \rho_1, \dots, \rho_k\}$, $\rho_i \neq I$, $i \geq 2$. Then every element in $T_{\mathbb{Z}} R_{\mathbb{Z}}$ can be written as $t\rho_i$ for some $i = 1, \dots, k$ and $t \in T_{\mathbb{Z}}$.

Now, let $p > 2$ be a prime such that

$$(a) \ p \text{ does not divide } e^2,$$

$$(b) \ \rho_i \not\equiv I \pmod{p}, i = 2, \dots, k,$$

$$(c) \ p \text{ does not divide any denominator of the entries of } B \text{ and } B^{-1}.$$

Then we claim that for such a prime p , $(T_{\mathbb{Z}} R_{\mathbb{Z}})_p = (T_{\mathbb{Z}})_p$.

To prove this, it is enough to prove $(T_{\mathbb{Z}} R_{\mathbb{Z}})_p \subset (T_{\mathbb{Z}})_p$. So, take $t\rho_i \in (T_{\mathbb{Z}} R_{\mathbb{Z}})_p$. Then $t\rho_i \equiv I \pmod{p}$. Hence $te = t\rho_i e \equiv e \pmod{p}$, which implies $te = e + pv$ for some $v \in \mathbb{Z}^{n+1}$. Recall that $t = t_a$, where $a = (a_1, \dots, a_{n-1})$ is such that

$te = a_1r_1 + \dots + a_{n-1}r_{n-1} + a_nx_0 + e$. Hence $pv = a_1r_1 + \dots + a_{n-1}r_{n-1} + a_nx_0$. But

$$pv = BB^{-1}pv = BB^{-1}(a_1r_1 + \dots + a_{n-1}r_{n-1} + a_nx_0) = B(a_1, \dots, a_n, 0)$$

so that $pB^{-1}v = (a_1, \dots, a_n, 0)$.

Consequently, because of (c) above, we have that $a = (a_1, \dots, a_{n-1})$ is such that p divides every numerator of the a_i 's and does not divide any of its denominators, and the same is true for the entries of the matrix A_a . Then, because of (c) again, we can see that

$$t = t_a = B(t_a)_B B^{-1} = B(I + A_a)B^{-1} = I + BA_a B^{-1} \equiv I \pmod{p}.$$

Step 3. We show now that we can find congruence subgroups of $G_{\mathbb{Z}}$ contained in $T_{\mathbb{Z}}R_{\mathbb{Z}}$. For this write

$$B = \frac{1}{s_1}B_1, \\ B^{-1} = \frac{1}{s_2}B_2,$$

where $s_i \in \mathbb{Z}$ and B_i has integer entries.

Take $m = 2s_1s_2^2$. Let $g \in (G_{\mathbb{Z}})_m$, that is, $g \equiv I \pmod{m}$. Because of step 1, $g = t_a\rho_i$ and then $t_a e = ge$. We show that $t_a, t_a^{-1} \in T_{\mathbb{Z}}$. Recall that $a = (a_1, \dots, a_{n-1})$ is such that $ge = a_1r_1 + \dots + a_{n-1}r_{n-1} + a_nx_0 + e$. As in step 2, $ge = e \pmod{m}$ implies $ge = e + mv$. Then

$$mv = BB^{-1}mv = BB^{-1}(a_1r_1 + \dots + a_{n-1}r_{n-1} + a_nx_0) = B(a_1, \dots, a_n, 0)$$

and we have $2s_1s_2B_2v = m\frac{1}{s_2}B_2v = mB^{-1}v = (a_0, \dots, a_n, 0)$.

Hence a has integer entries. Moreover, we have that s_1s_2 divides all the entries of a so that the same happens with the entries of A_a and A_{-a} . Then

$$t_a = B(t_a)_B B^{-1} = B(I + A_a)B^{-1} = I + BA_a B^{-1} = I + B_1 \left[\frac{1}{s_1s_2}A_a \right] B_2 \in T_{\mathbb{Z}}$$

and analogously for $t_a^{-1} = B(I + A_{-a})B^{-1}$.

This proves step 3 because

$$g = t_a(t_a^{-1}g) \in T_{\mathbb{Z}}R_{\mathbb{Z}}.$$

Step 4. We complete the proof. Use step 3 to get $(G_{\mathbb{Z}})_m$ contained in $T_{\mathbb{Z}}R_{\mathbb{Z}}$. Take p coprime to m and apply step 2 to get $(G_{\mathbb{Z}})_{mp} = ((G_{\mathbb{Z}})_m)_p \subset (T_{\mathbb{Z}}R_{\mathbb{Z}})_p = (T_{\mathbb{Z}})_p$. (To see that $T_{\mathbb{Z}} \cong \mathbb{Z}^{n-1}$, note that $T_{\mathbb{Z}}$ is a discrete cocompact subgroup of $T \cong \mathbb{R}^{n-1}$.)

7.2.1. Remark. Note that step 4 above implies $(G_{\mathbb{Z}})_{mp} \subset (T_{\mathbb{Z}})_{mp}$. From this and $T \subset G$, it follows that $(G_{\mathbb{Z}})_{mp} = (T_{\mathbb{Z}})_{mp}$, for all but finite primes p . Also note that if $a \in \mathbb{Z}$ is such that $(G_{\mathbb{Z}})_a = (T_{\mathbb{Z}})_a$, then, for every $c \in \mathbb{Z}$, we have $(G_{\mathbb{Z}})_{ac} = (T_{\mathbb{Z}})_{ac}$. (This is because $(G_{\mathbb{Z}})_a = (T_{\mathbb{Z}})_a$ implies $(G_{\mathbb{Z}})_{ac} = ((G_{\mathbb{Z}})_a)_{ac} = ((T_{\mathbb{Z}})_a)_{ac} = (T_{\mathbb{Z}})_{ac}$.)

7.3. Corollary. $M(\mathcal{Q}, a)$ has a finite cover $M(\mathcal{Q}, ba)$, with all cusps diffeomorphic to $T^{n-1} \times (0, \infty)$.

Proof. Recall that to an isotropic rational vector x corresponds a cusp. But different rational isotropic vectors may correspond to the same cusp. In fact, if x determines a cusp, then the set of all isotropic rational vectors determining the same cusp is $\{gx : g \in \Gamma\}$, where $\Gamma = (O(\mathcal{Q})_{\mathbb{Z}})_a$ is the fundamental group of the manifold $M(\mathcal{Q}, a)$.

The problem is that after taking finite covers, x and gx may not correspond to the same cusp anymore. Given $y \in \mathbb{R}^{n+1}$, let $G^y = \{g \in O(\mathcal{Q}) : gy = y\}$. Because we have a finite number of cusps, what we have to prove is that if a is such that $((G^x)_{\mathbb{Z}})_a \cong \mathbb{Z}^{n-1}$, then $((G^{gx})_{\mathbb{Z}})_a \cong \mathbb{Z}^{n-1}$, for $g \in \mathcal{O}(Q)_{\mathbb{Z}}$. For this just note that $g^{-1}G^{gx}g = G^x$. Thus, if $g \in \mathcal{O}(Q)_{\mathbb{Z}}$, then $g^{-1}((G^{gx})_{\mathbb{Z}})_a g \subset ((G^x)_{\mathbb{Z}})_a \cong \mathbb{Z}^{n-1}$. The corollary follows.

7.4. Corollary. *Suppose that $M(\mathcal{Q}, a)$ has a cusp diffeomorphic to $T^{n-1} \times [0, \infty)$. Then there are integers b and q_0 such that if $q \geq q_0$, $(q, ba) = 1$, we have that $M(\mathcal{Q}, qba)$ has a large cusp.*

Proof. We use all notation from the proof of theorem 5. Recall that we have a map $t : \mathbb{R}^{n-1} \rightarrow G$, $a \mapsto t_a$.

Claim. If c is such that $t^{-1}((T_{\mathbb{Z}})_c) \subset \mathbb{Z}^{n-1}$, then $t^{-1}[(T_{\mathbb{Z}})_{cq}] = t^{-1}[(T_{\mathbb{Z}})_c] \cap [q\mathbb{Z}]^{n-1}$, where q is large enough and coprime to c .

Proof of the claim. If $a \in t^{-1}[(T_{\mathbb{Z}})_{cq}]$, then $t_a \in (T_{\mathbb{Z}})_{cq}$ and we have $t_a \in (T_{\mathbb{Z}})_c$. We have to prove that $a \in [q\mathbb{Z}]^{n-1}$. For this, note that we also have $t_a \in (T_{\mathbb{Z}})_q$, which implies $BA_aB^{-1} \equiv 0 \pmod{q}$, and, by taking q large enough (so that q does not divide any denominator of the entries of B and B^{-1}), we have that q divides every entry of A_a , so that q divides every entry of a .

Conversely, assume $a \in t^{-1}[(T_{\mathbb{Z}})_c] \cap [q\mathbb{Z}]^{n-1}$. Then $t_a \in (T_{\mathbb{Z}})_c$. On the other hand, because $a \in [q\mathbb{Z}]^{n-1}$, $A_a \equiv 0 \pmod{q}$ and proceeding as before, we get $t_a \equiv I \pmod{q}$. But $(q, c) = 1$, and hence from $t_a \equiv I \pmod{c}$ and $t_a \equiv I \pmod{q}$, we get $t_a \equiv I \pmod{cq}$. This proves the claim.

Recall from remark 7.2.1 that there is a b such that $(G_{\mathbb{Z}})_{ba} = (T_{\mathbb{Z}})_{ba}$. Also, it is not difficult to see that we can take b such that $t^{-1}((T_{\mathbb{Z}})_b) \subset \mathbb{Z}^{n-1}$. These observations together with the claim imply (take $c = ba$ in the claim) $t^{-1}[(G_{\mathbb{Z}})_{(ba)q}] = t^{-1}[(G_{\mathbb{Z}})_{(ba)}] \cap [q\mathbb{Z}]^{n-1}$, where $(ba, q) = 1$.

Now, if $\{v_1, \dots, v_{n-1}\}$ is a basis for $t^{-1}[(G_{\mathbb{Z}})_{(ba)}]$, we claim that $\{qv_1, \dots, qv_{n-1}\}$ is a basis for $t^{-1}[(G_{\mathbb{Z}})_{(ba)q}]$. For this take $q > \det E$, where E is the matrix whose columns are the (integer) coordinates of the v_i 's. Then $\det E \not\equiv 0 \pmod{q}$. Now if $x \in t^{-1}[(G_{\mathbb{Z}})_{(ba)}] \cap [q\mathbb{Z}]^{n-1}$, then $x = \sum \lambda_i v_i$ and we have to prove that q divides all λ_i 's. But $x \in [q\mathbb{Z}]^{n-1}$, and hence $E\lambda \equiv 0 \pmod{q}$, where $\lambda = (\lambda_1, \dots, \lambda_{n-1})$, and because $\det E \not\equiv 0 \pmod{q}$ E has an inverse in \mathbb{Z}_q (q is prime) and we get $\lambda \equiv 0 \pmod{q}$.

This means that the coverings obtained are simple (with respect to $\{v_1, \dots, v_{n-1}\}$) and, because we can take q as large as we want, the corollary follows from lemma 2.14.

Let $T^1 = \{t_{se_1} : s \in \mathbb{R}\}$, that is, T_1 is the subgroup of T generated by t_{e_1} . Also, let $T^2 = \{t_y : y \in \{0\} \times \mathbb{R}^{n-2}\}$.

7.5. Lemma. *If the isotropic vector x_0 has a zero first coordinate (i.e., $x_0 = (0, *, *, \dots, *)$) and \mathcal{Q} is diagonal of the form $\lambda_1 x_1^2 + \dots + \lambda_n x_n^2 - \lambda_{n+1} x_{n+1}^2$, $\lambda_i > 0$, then, for any integer $k \neq 0$, we have*

$$(T_{\mathbb{Z}})_{2k} = (T_{\mathbb{Z}}^1)_{2k} \oplus (T_{\mathbb{Z}}^2)_{2k}.$$

Proof. We will use the notation of the proof of lemma 7.2. Because the first coordinate of x_0 is zero, we can take $r_1 = e_1 = (1, 0, \dots, 0)$, and r_i , $r \geq 2$ and e having

their first coordinates equal to zero. Then the matrix B , whose columns are the $r_1, \dots, r_{n-1}, x_0, e$ can be written as $\begin{pmatrix} 1 & 0 \\ 0 & B_0 \end{pmatrix}$, where the columns of B_0 are the $\{r_2, \dots, r_{n-1}, x_0, e\}$, with the first coordinates (which are zero) deleted.

Let $a \in \mathbb{R}^{n-1}$. Then $a = a_1 e_1 + b$, $a_1 \in \mathbb{R}$, $b \in \{0\} \times \mathbb{R}^{n-2}$. Thus $t_a = t_{a_1 e_1} t_b$ (recall that $a \mapsto t_a$ is a homomorphism). Now, for any integer l , we have $t_a \equiv I \pmod{l}$ iff $I + BA_a B^{-1} \equiv I \pmod{l}$ iff $BA_a B^{-1} \equiv 0 \pmod{l}$. But $B = \begin{pmatrix} 1 & 0 \\ 0 & B_0 \end{pmatrix}$ and $A_a = A_{a_1 e_1} + A_b$, and a straightforward calculation shows

$$BA_a B^{-1} = \begin{pmatrix} 0 & a_1 y \\ a_1 e_1^2 x_0 & B_0 A_b B_0^{-1} + C \end{pmatrix}$$

where y is the last line of B_0^{-1} and $C = (\frac{1}{2} a_1^2 e_1^2) x_0 y$ (note that x_0 is an $n \times 1$ matrix and y is a $1 \times n$ matrix).

Hence, if $BA_a B^{-1} \equiv 0 \pmod{l}$, then $a_1 e_1^2 x_0 \equiv 0 \pmod{l}$ and $a_1 y \equiv 0 \pmod{l}$. This implies, together with $C = (\frac{1}{2} a_1^2 e_1^2) x_0 y$ and $l = 2k$, that $C \equiv 0 \pmod{2k}$ and $B_0 A_b B_0^{-1} \equiv 0 \pmod{2k}$. Conversely, if $B_0 A_b B_0^{-1} \equiv 0 \pmod{2k}$, $a_1 e_1^2 x_0 \equiv 0 \pmod{2k}$ and $a_1 y \equiv 0 \pmod{2k}$, we get $BA_a B^{-1} \equiv 0 \pmod{2k}$. Consequently, we have the equivalence $BA_a B^{-1} \equiv 0 \pmod{2k}$ iff $BA_{a_1 e_1} B^{-1} \equiv 0 \pmod{2k}$ and $BA_b B^{-1} \equiv 0 \pmod{2k}$. Equivalently, we get $t_a \equiv I \pmod{2k}$ iff $t_{a_1 e_1} \equiv I \pmod{2k}$ and $t_b \equiv I \pmod{2k}$. This proves the lemma.

Let $J = \text{diag}\{-1, 1, 1, \dots, 1\}$. The proof of the following lemma is a straightforward calculation.

7.6. Lemma. *Consider the involution $\iota : T_{\mathbb{Z}} \rightarrow T_{\mathbb{Z}}$, given by $\iota(g) = JgJ$. Then, under the same assumptions as lemma 7.5, we have*

- (i) $\text{fix}(\iota) = T_{\mathbb{Z}}^2$,
- (ii) $\iota(t_{e_1}) = t_{-e_1} = t_{e_1}^{-1}$.

Note that $g \equiv I \pmod{a}$ iff $\iota(g) \equiv I \pmod{a}$. Hence, we can consider $\iota : (G_{\mathbb{Z}})_a \rightarrow (G_{\mathbb{Z}})_a$, for any $a \in \mathbb{Z}$. The next corollary follows from lemmas 7.5 and 7.6, remark 7.2.1 and 4.6.1.

7.8. Corollary. *Under the same assumptions as lemma 7.5, we have that there is $a \in \mathbb{Z}$, such that for all $c \in \mathbb{Z}$, $\iota : (G_{\mathbb{Z}})_{ac} \rightarrow (G_{\mathbb{Z}})_{ac}$ is simple.*

8. PROOF OF THEOREM 6

Theorem 6. *Let $M(\mathcal{Q}, a)$ be as before with \mathcal{Q} satisfying 0.1. Then $M(\mathcal{Q}, a)$ admits finite covers M_0 , such that there are non-compact finite volume real hyperbolic manifolds M_1, M_2 satisfying the following conditions:*

- (1) $\dim M_0 \geq 5$, $\dim M_1 = n - 1$, $\dim M_2 = n - 3$;
- (2) $M_2 \subset M_1 \subset M_0$ and the inclusions are totally geodesic;
- (3) the normal bundle of M_2 in M_0 is trivial;
- (4) M_0 has a large cusp C_{M_0} diffeomorphic to $T^{n-1} \times (0, \infty)$;
- (5) M_i has exactly one cusp contained in C_{M_0} , $i = 1, 2$.

Proof. Let \mathcal{Q} be a diagonal rational quadratic form of signature $(n, 1)$,

$$Q(x_1, \dots, x_{n+1}) = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2 - \lambda_{n+1} x_{n+1}^2, \quad \lambda_i \in \mathbb{Q}$$

with $\lambda_i > 0$, and $\lambda_1 = \lambda_2 = \lambda_3$.

By (0.1), if $n \leq 6$, $\lambda_4 x_4^2 + \dots + \lambda_n x_n^2 - \lambda_{n+1} x_{n+1}^2$ has a rational isotropic vector. Denote this vector by $(0, 0, 0, z_4, \dots, z_n, z_{n+1})$.

If $n \geq 7$, the form

$$\lambda_4 x_4^2 + \dots + \lambda_{n-2} x_{n-2}^2 + \lambda_{n-1} x_{n-1}^2 + \lambda_n x_n^2 - \lambda_{n+1} x_{n+1}^2$$

has, by the Hasse-Minkowski Theorem, a rational isotropic vector. Denote this vector also by $(0, 0, 0, z_4, \dots, z_n, z_{n+1})$. In any case, our rational isotropic vector has, at least, the first three coordinates equal to zero.

Now, $M(\mathcal{Q}, a)$ (for all but a finite number of primes a) has a cusp C_M corresponding to that isotropic vector (see lemma 7.2).

Because of theorem 5 we can assume that C is diffeomorphic to $T^{n-1} \times (0, \infty)$.

Now, by [14], $M(\mathcal{Q}, a)$ has a finite orientable cover $M = M(\mathcal{Q}, ba)$ with three totally geodesic orientable hypersurfaces H_i , $i = 1, 2, 3$, that do not separate M and are contained in the fixed point set of an isometric involution ι_i . In fact, if $p_M : \mathbb{I}^n \rightarrow M$ is the covering projection ($\mathbb{I}^n \subset \mathbb{R}^{n+1}$ is the hyperboloid model of the hyperbolic space \mathbb{H}^n), then $H_i = p_M(\{x_i = 0\})$, where $\{x_i = 0\} = \mathbb{I}^n \cap \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i = 0\}$. (To obtain H_i , $i = 1, 2, 3$, apply [14] to each involution $\iota_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, $\iota_i(x_1, \dots, x_i, \dots, x_{n+1}) = (x_1, \dots, -x_i, \dots, x_{n+1})$ fixing $\{x_i = 0\}$. For each $i = 1, 2, 3$, [14] gives a congruence subgroup. Take the intersection.) Note that the H_i 's are mutually orthogonal.

By [14] again we can assume also that $N = p_M(\{x_1 = x_2 = x_3 = 0\}) \subset H_1 \cap H_2 \cap H_3$ is a totally geodesic submanifold of M . (Here note that $\iota(x_1, x_2, x_3, x_4, \dots, x_{n+1}) = (-x_1, -x_2, -x_3, x_4, \dots, x_{n+1})$ fixes $\{x_1 = x_2 = x_3 = 0\}$ and use lemma 2.1 of [14].) We have $N \subset H_i \subset M$, $i = 1, 2, 3$, and all the inclusions are totally geodesic.

Then the H_i 's and N are non-compact and have a cusp corresponding to the same rational isotropic vector $(0, 0, 0, z_4, \dots, z_{n-2}, z_{n-1}, z_n, z_{n+1})$. Hence H_i , $i = 1, 2, 3$, all have at least one cusp C_{H_i} contained in C_M . By corollary 7.4, we can assume C_M to be large. (Note that the properties of $p_M(\{x_i = 0\})$ being a manifold, being contained in the fixed point set of an isometric involution, not separating M , and having the cusp corresponding to $(0, 0, 0, z_4, \dots, z_n, z_{n+1})$ with torus cross section, are not destroyed by taking further congruence subgroups.)

8.1. Note that, because $\lambda_i > 0$ and $\lambda_1 = \lambda_2 = \lambda_3$, we have that, for $i, j \in \{1, 2, 3\}$, $i \neq j$, there is an isometry $f : M \rightarrow M$, with $fH_i = H_j$. (For example, for $i = 1$, $j = 2$, we can take f induced by $(x_1, x_2, x_3, \dots, x_{n+1}) \mapsto (x_2, x_1, x_3, \dots, x_{n+1})$.)

Let T_M denote the flat torus cross section of C_M , and consider the restriction $\bar{\iota}_i|_{T_M} : T_M \rightarrow T_M$, of the isometric involution $\bar{\iota}_i : M \rightarrow M$, induced by $\iota_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, defined above. (Note that H_i is contained in the fixed point set of an involution, and this involution sends C_M to itself; hence, the involution induces an isometric involution on the cross section T_M of the cusp C_M .) Then corollary 7.8 implies that we can suppose $\bar{\iota}_i$ is simple (note that, for instance, $\bar{\iota}_1 : M \rightarrow M$ at the level of matrices is given by conjugation with the matrix $J = \text{diag}\{-1, 1, \dots, 1\}$).

We have two cases.

First case. Suppose H_1 has exactly one cusp in C_M .

In this case, by 8.1, the same happens with H_2 and H_3 . Let T_{H_i} be the corresponding tori cross sections of the (only) cusps of H_i contained in C_M . Again by 8.1, and the fact that the isometries induced by the ι_i are simple, we get that each T_{H_i} is simple in T_M . Hence, from the definition of simple subtorus and lemma 4.7,

we get that $T_M \cap N \subset \bigcap_{i=1}^3 T_{H_i}$ is connected. Hence, N has exactly one cusp in C_M . This proves the theorem in this case.

Second case. Suppose H_1 has more than one cusp in C_M .

Recall that H_i is contained in the fixed point set of an involution. Hence, by lemma 4.6, H_1 has exactly two cusps in C_M .

Let α_i be the element in $\pi_1(M)$ representing the orthogonal \mathbb{S}^1 to T_{H_i} (see 4.6). Because $\pi_1(M)$ is residually finite, there is a homomorphism from $\pi_1(M)$ to a finite group such that the image of each α_i has order larger than two, hence is not zero (in our case this is simple, for we can take a prime p not dividing one of the non-diagonal entries of the integer matrix corresponding to α_i and take congruence mod p). Take the finite cover corresponding to the kernel of this homomorphism and call this cover M' and choose a cusp $C_{M'}$ that maps to C_M by the covering projection (note that we can choose p to be very large so that $C_{M'}$ is still large; see corollary 7.4). Now, the pre-image of C_{H_i} by the covering projection intersected with $C_{M'}$ gives many cusps (more than one because we are unwrapping all α_i 's). Let $H'_i = p_{M'}(\{x_i = 0\})$, so that H'_i maps to H_i by the covering projection. If one H'_i has exactly one cusp in $C_{M'}$, by the first case, we are done. So, we assume that H_1 (hence H_i , $i = 1, 2, 3$) has more than one cusp in $C_{M'}$. Recall that we can still suppose that H'_i is in the fixed point set of an isometric involution of M' , so that H'_i also has exactly two cusps in $C_{M'}$. Note that the pre-image of C_{H_i} by the covering projection intersected with $C_{M'}$ has more than one component so that there is another connected H''_i , whose image is also H_i by the covering projection, that has a cusp in $C_{M'}$ (that maps to C_{H_i}). But the covering is normal (congruence subgroups are normal subgroups) so that the covering transformations are transitive in the fibers, which implies that there is a covering transformation sending H'_i to H''_i . Hence H''_i has also two cusps in $C_{M'}$. Note that H'_i and H''_i do not intersect.

Now, we construct, for each $i = 1, 2, 3$, the double cover corresponding to H''_i . Since H'_i (and then H''_i) does not separate M' , there is a closed path β_i that intersects H''_i in one point, which means that it is not zero in $H_1(M', \mathbb{Z}_2)$ so that the composite

$$\pi_1(M') \rightarrow H_1(M', \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$$

(the last map sends β_i to one) gives a non-zero homomorphism whose kernel has index two, which corresponds to a double cover. We can also do this geometrically. Given $i = 1, 2, 3$, take two copies of M' , $M'_i(1), M'_i(2)$. Cut along $H''_i(j)$ in $M'_i(j)$, $j=1, 2$, so that we end up with four copies of $H''_i : H''_i(1)^+$ and $H''_i(1)^-$ from $M'_i(1)$ and $H''_i(2)^+$ and $H''_i(2)^-$ from $M'_i(2)$. Identify $H''_i(1)^+$ with $H''_i(2)^-$ and $H''_i(1)^-$ with $H''_i(2)^+$. These are the double covers M''_i of M' corresponding to H''_i (note that it is connected because H''_i does not separate M') and since H'_i does not intersect H''_i , the pre-image of H'_i by this double cover is just two disjoint copies of H'_i . Note also that if we take one point in each cusp of H'_i inside $C_{M'}$ we can join these two points by one path in H'_i (so that it does not intersect H''_i) and we can join them also by another path inside $C_{M'}$ that intersects H''_i once. By joining these two paths, we obtain a loop that intersects H''_i once so that its lifting begins in one cusp and ends in another. This proves that any pre-image of H'_i has just one cusp in a cusp of the double cover constructed. Note that, because we are not unwrapping any loop in the cusp $C_{M'}$, we still have that every pre-image of $C_{M'}$ is still large. (In fact, we have two pre-images of $C_{M'}$ each isometric to $C_{M'}$, which in turn implies $\pi_1(C_{M'}) \subset \pi_1(M''_i)$.)

Let $\Gamma = (\pi_1 M_1'') \cap (\pi_1 M_2'') \cap (\pi_1 M_3'') \subset \pi_1 M'$. Then, Γ is of finite index in $\pi_1 M'$, and $\pi_1 C_{M'} \subset \Gamma$ (because $\pi_1 C_{M'} \subset \pi_1 M_i''$, $i = 1, 2, 3$). Let $M_0 = \mathbb{H}^n / \Gamma$, $M_1^i = p_{M_0}(\{x_i = 0\})$, $i = 1, 2, 3$ and $M_2 = p_{M_0}(\{x_1 = x_2 = x_3 = 0\})$. We have that $M_1 = M_1^1$ has exactly one cusp in C_{M_0} , with cross section $T_{M_1^1}$; and, because the projection map $C_{M_0} \rightarrow C_{M'}$ is an isometry, each $T_{M_1^i}$ is simple in C_{M_0} (recall that each $T_{M_i^i}$ is simple in $T_{M'}$). Then, lemma 4.7 implies that M_2 has exactly one cusp in M_0 . This completes the proof.

9. PROOF OF THEOREM 2

Here we prove theorem 2, but first we need some notation and remarks. Let M be a non-compact finite volume real hyperbolic manifold, with cusps $F_i \times [0, \infty)$, $i = 1, \dots, k$, where the F_i 's are flat manifolds. Consider, for each i , the totally geodesic embedding $F_i \subset DM$, where the curvature outside $\bigcup F_i$ is strictly negative (see section 1).

Note now that \mathbb{R}^n is (diffeomorphic to) the universal cover of DM , and call the covering projection p . Then $p^{-1}(\bigcup F_i)$ is a disjoint union of closed hypersurfaces of \mathbb{R}^n , each diffeomorphic to \mathbb{R}^{n-1} ; hence each divides \mathbb{R}^n into two components having the hypersurface as common boundary. Denote by \mathcal{W} the set of all these hypersurfaces (then $\bigcup \mathcal{W} = p^{-1}(\bigcup F_i)$) and call each element of \mathcal{W} a wall.

Recall that we have two canonical embeddings of $M^0 = M \setminus \bigcup \{F_i \times (b, \infty)\}$, for some b , in DM , $j_i : M^0 \rightarrow DM$, $i = 1, 2$, and $j_1(M^0) \cap j_2(M^0) = j_1(\partial M^0) = j_2(\partial M^0) = \bigcup F_i$. Then $p^{-1}(j_i(\text{int } M^0))$, $i = 1, 2$, is a disjoint union of open subsets of \mathbb{R}^n , all diffeomorphic to \mathbb{R}^n , with boundary a union of walls. Call them regions. Note that each wall is in the boundary of exactly two regions, one in $p^{-1}(j_1(M^0))$ and the other in $p^{-1}(j_2(M^0))$.

Choose now $x_i \in F_i \subset DM$ and $\tilde{x}_i \in \mathbb{R}^n$ such that $p(\tilde{x}_i) = x_i$, and this gives a correspondence (one for each i) between $\pi_1(DM, x_i)$ and the deck transformations on \mathbb{R}^n . Under this correspondence we have the following:

(9.1)

$$g \in \pi_1(F_i, x_i) \subset \pi_1(DM, x_i) \text{ if and only if } gW_i = W_i, \quad \text{for } i = 1, \dots, k.$$

where W_i is the wall mapping to F_i and containing \tilde{x}_i . We now have a lemma.

9.2. Lemma. *Let g be a deck transformation such that $(g^\ell)W = W$ for some wall W and some integer ℓ . Then $gW = W$.*

Proof. First we have a claim.

Claim. Let $z : \mathbb{R} \rightarrow \mathbb{R}^n$ be a geodesic (\mathbb{R}^n with the metric pulled back from DM) contained in a wall W . Then $d(z(t), W_0) \rightarrow \infty$, as $t \rightarrow \infty$, for $W_0 \neq W$.

Proof of the claim. Choose a point $x \in W_0$. Then if z' is a geodesic passing through x and contained in W_0 , [13] implies that $\inf\{d(z(t), z'(r)); r \in \mathbb{R}\} \rightarrow \infty$, as $t \rightarrow \infty$ (there is no flat curvature outside the walls). But the set of all geodesic rays in W_0 beginning at x is parametrized by the sphere S^{n-1} , which is compact, so that the claim follows because every point in W_0 can be joined to x by a geodesic.

Define now $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by $f(p) = \sum_{j=1}^{\ell} d(p, (g^j)W)$. Then f is continuous and because the distance between walls goes to infinity as we approach infinity (this follows from the claim above) f has a minimum value at a point p_0 . Now, if $gW \neq W$, then p_0 is unique (unique because f is now strictly convex, i.e., $\frac{d^2 f \gamma(t)}{dt^2} > 0$, for

all geodesics γ). But $f(gp_0) = \sum_{j=1}^{\ell} d(gp_0, (g^j)W) = \sum_{j=1}^{\ell} d(p_0, (g^j)W) = f(p_0)$, and hence $p_0 = gp_0$, a contradiction because g acts freely. This completes the proof of lemma 9.2.

Because the F_i are flat manifolds, each F_i has a flat torus finite cover $q_i : T_i^{n-1} \rightarrow F_i$. Denote the image of $\pi_1(T_i^{n-1}) \cong \mathbb{Z}^{n-1}$ in $\pi_1(DM, x_i)$ by the map $\pi_1(T_i^{n-1}) \xrightarrow{(q_i)^*} \pi_1(F_i, x_i) \rightarrow \pi_1(DM, x_i)$ by Z_i . Recall that there is curvature zero only at the F_i 's; hence, because of the torus theorem of [13] and because $n > 2$, the only subgroups of $\pi_1(DM, *)$ isomorphic to \mathbb{Z}^{n-1} are the conjugates of the $\beta^{-1}Z_i\beta$'s, where β is a path joining base points. Note that, for $i \neq j$, Z_i and Z_j do not induce conjugate subgroups. (Proof: if they were conjugate, the immersions $T^{n-1} \rightarrow F_i \rightarrow DM$ and $T^{n-1} \rightarrow F_j \rightarrow DM$ would be homotopic (because F_i, F_j and DM are aspherical) and their lifts would be at bounded distance. Hence, by the claim above, they would be equal.)

We now prove theorem 2.

Theorem 2. *Let $f : DM \rightarrow DM$ be a homeomorphism, where M is a non-compact finite volume real hyperbolic n -manifold, $n > 2$. Then $f \sim g$, where $g : DM \rightarrow DM$ is a diffeomorphism.*

Proof. Let $f : DM \rightarrow DM$ be a homeomorphism and let $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a lifting, that is, $f p = p \tilde{f}$. We have that $f_* : \pi_1(DM, *) \rightarrow \pi_1(DM, f(*))$ permutes the conjugacy classes corresponding to the Z_i 's. Hence we get a bijection $f : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ such that $f_* Z_i = \beta^{-1} Z_{f(i)} \beta$, where β is a certain path joining $f(x_i)$ and $x_{f(i)}$. By moving f by an isotopy, we can suppose then that $f(x_i) = x_{f(i)}$. Moreover, we can suppose that $f_* Z_i = Z_{f(i)}$, for we could obtain any conjugation by moving the base point $x_{f(i)}$ around a closed path, again by an isotopy.

9.3. Claim. $f_*(\pi_1(F_i, x_i)) = \pi_1(F_{f(i)}, x_{f(i)})$.

To see this, take $g \in \pi_1(F_i, x_i)$. Because Z_i has finite index in $\pi_1(F_i, x_i)$, we have that $g^\ell \in Z_i$ for some integer ℓ . Then $f_*(g^\ell) \in Z_{f(i)} \subset \pi_1(F_{f(i)}, x_{f(i)})$, and, by remark 9.1, $f_*(g^\ell)W_{f(i)} = W_{f(i)}$. But $f_*(g^\ell) = (f_*g)^\ell$ and lemma 9.2 implies $(f_*g)W_{f(i)} = W_{f(i)}$. Hence, by 9.1 again, $f_*g \in \pi_1(F_{f(i)}, x_{f(i)})$. This proves one inclusion and, proceeding in the same way with f^{-1} , we get the other inclusion. This proves claim 9.3. Note that this result proves, in particular, that F_i and $F_{f(i)}$ are diffeomorphic (flat manifolds are differentiably rigid).

Now, denote $W \cap p^{-1}(\{x_1, \dots, x_k\})$ by L_W , where W is a wall, and write also $L = \bigcup L_W = p^{-1}(\{x_1, \dots, x_k\})$. Now, because $f(x_i) = x_{f(i)}$ and $f_*(\pi_1(F_i, x_i)) = \pi_1(F_{f(i)}, x_{f(i)})$, we have that for every wall W there is a wall W' such that $\tilde{f}(L_W) = L_{W'}$. This gives a bijection f^W from W to itself.

9.4. Claim. If W_1 and W_2 are walls of a region R (i.e., $W_1, W_2 \subset \partial R$), then $W'_1 = f^W(W_1)$ and $W'_2 = f^W(W_2)$ are also walls of a region R' .

The proof of 9.4 is given in two steps.

Step 1. Let W be a wall and $W' = f^W(W)$. We prove that there is a bounded homotopy $H : W \times [0, 1] \rightarrow \mathbb{R}^{n+1}$, such that $H_0 = \tilde{f}|_W$ and $H_1 : W \times \{1\} \rightarrow W'$ is a diffeomorphism (a bounded homotopy means that there is a constant s such that $\forall x \in W$, $\text{diam}\{H(x, [0, 1])\} \leq s$). To see this, note that because DM and F_i are aspherical and $f_*(\pi_1(F_i, x_i)) = \pi_1(F_{f(i)}, x_{f(i)})$, we have that the map $f|_{F_i} : F_i \rightarrow DM$ is homotopic, by a homotopy H' , to a diffeomorphism $h' : F_i \rightarrow F_{f(i)} \subset DM$

with $h'_* = f_*|_{\pi_1(F_i)}$. Then lifting H' and h' , we obtain H and h , and because DM is compact, H' and H are bounded.

Step 2. We complete the proof. Take W_1 and W_2 walls of a region R . This is equivalent to saying that there is no wall separating them. Write $W'_i = f^{\mathcal{W}}(W_i)$, $i = 1, 2$. If W'_1 and W'_2 are not walls of a region, then there is a wall $W'_0 = f^{\mathcal{W}}(W_0)$ separating them. By step 1, $\tilde{f}|_{W_0}$ can be deformed to a diffeomorphism $h : W_0 \rightarrow W'_0$ by a homotopy of finite diameter s . Let $z_i \in L_{W_i}$ be such that for $y_i = \tilde{f}(z_i) \in L_{W'_i}$, $i = 1, 2$, we have $d(y_i, W'_0) > s$ (we are using the claim in lemma 9.2 here). Then, because W_0 does not separate z_1 and z_2 , $\tilde{f}(W_0)$ does not separate $\tilde{f}(z_1) = y_1$ and $\tilde{f}(z_2) = y_2$. But then, because W'_0 and $\tilde{f}(W_0)$ are s -closed and $d(y_i, W'_0) > s$, we get that W'_0 also does not separate y_1 and y_2 , which is a contradiction.

Then we can join y_1 and y_2 by a path γ that does not intersect $\tilde{f}(W_0)$ (W_0 does not separate W_1 and W_2) and $f\gamma$ is homotopic rel $\{0,1\}$ to a path γ' that intersects W'_0 once (W'_0 does separate W'_1 and W'_2). This is a contradiction since the bounded homotopy between $\tilde{f}(W_0)$ and W'_0 does not pass through y_1, y_2 , and $\tilde{f}\gamma \cap \tilde{f}(W_0) = \emptyset$. This completes the proof of claim 9.4. We now complete the proof of theorem 2.

First we see that claim 9.4 implies

9.5. $f_*((j_i)_*(\pi_1(M^0))) = (j_{i'})_*(\pi_1(M^0))$, where we have either $i = i'$ or $i \neq i'$ for $i, i' = 1, 2$.

To see this take, for instance, a closed path $\gamma \in \pi_1(j_1(M^0))$. Then lifting γ we obtain a path $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{R}^n$ that joins two walls and such that $\tilde{\gamma}(0, 1)$ does not intersect any walls. Claim 9.4 implies $\tilde{f}(\tilde{\gamma}(0))$ and $\tilde{f}(\tilde{\gamma}(1))$ lie on walls of a region R' , so that they can be joined by a path $\alpha : [0, 1] \rightarrow \mathbb{R}^n$ with $\alpha(0, 1) \subset R'$. But $\alpha \sim \tilde{f}\tilde{\gamma}$ rel $\{0, 1\}$, which means that $f_*[\tilde{\gamma}]$ belongs to $(j_i)_*\pi_1(M^0)$ for i such that $p(R') = j_i(M^0)$. This proves 9.5.

Write $\sigma_i = f_*|_{\pi_1(j_i(M^0))} : \pi_1(j_i(M^0)) \rightarrow \pi_1(j_{i'}(M^0))$. By Mostow's rigidity theorem, there are isometries $g_i : M \rightarrow M$, $i = 1, 2$, inducing (up to conjugation) the σ_i 's. But isometries send cusps to cusps and because any isometry of $F_i \times [0, \infty)$, with metric $e^{-2t}S + dt^2$, is clear to preserve, eventually, the t -levels (the radius of injectivity is a strictly decreasing function of t), we see that we can glue g_1 and g_2 obtaining a diffeomorphism $g : DM \rightarrow DM$, such that $f_* = g_*$ (note that conjugation is not a problem: just move the base point around the conjugating path using an isotopy that leaves the far end of the cusps invariant). But DM is aspherical, and hence $f \sim g$. This completes the proof of theorem 2.

Proof of theorem 2A. By [8] f is homotopic to a homeomorphism (for dimensions greater than four). The proof of theorem 2 above now follows word by word.

9.6. Corollary. *Let Σ be a PL structure on DM , $\dim M > 4$, non-concordant to the canonical one. Then (DM, Σ) is not PL equivalent to DM .*

Proof. Let $f : (DM, \Sigma) \rightarrow DM$ be a PL homeomorphism. We have two cases:

Particular Case. Suppose f is homotopic to the identity. Denote by H such homotopy and write $H : f \sim Id_{DM}$. Then $\overline{H} : f \times Id_{[0,1]} \sim Id_{DM \times [0,1]}$, where $\overline{H} = H \times Id_{[0,1]}$.

Since \overline{H} restricts to a homeomorphism on the boundary of $DM \times [0, 1]$, we may apply (1.6.1) of [8] to get a homotopy (which is constant on $\partial(DM \times [0, 1])$) of \overline{H} to a homeomorphism $\tilde{H} : DM \times [0, 1] \rightarrow DM \times [0, 1]$.

Because $\tilde{H}_1 = Id$ and $\tilde{H}_0 = f$, by pulling back the triangulation of $DM \times [0, 1]$ using \tilde{H} , we obtain a concordance between τ and τ_0 .

General Case. The general case follows by applying theorem 2 to $fg^{-1} \sim Id_{DM}$.

APPENDIX

We give a proof of lemma 2.4:

2.4. Lemma. *Let A be a Riemannian metric on $M \times I$ satisfying (2.1) and f as in (2.3). Then given $\epsilon > 0$, there is an L such that, for $\alpha > L$,*

$$-k_0 - \epsilon < K(A_\alpha) < -k_1 + \epsilon.$$

Proof. First we prove the following.

a.1. Claim. If A is any metric on $M \times I$ satisfying (2.1), then for any $(x_0, t_0) \in M \times I$, there is a coordinate system $(x_1^\alpha, \dots, x_n^\alpha)$ for M near x_0 and a linear coordinate t^α for $[a, b]$ near t_0 , such that

- (a) $A_\alpha = \sum g_{ij}^\alpha dx_i^\alpha dx_j^\alpha + d(t^\alpha)^2$ where $g_{ij}^\alpha = A_\alpha(\frac{\partial}{\partial x_i^\alpha}, \frac{\partial}{\partial x_j^\alpha})$.
- (b) $g^\alpha(x_0, t_0) = I$ where I is the identity matrix.
- (c) $\lim_{\alpha \rightarrow \infty} \frac{\partial g^\alpha}{\partial t^\alpha}(x_0, t_0) = 2\ell_1 I$,
 $\lim_{\alpha \rightarrow \infty} \frac{\partial^2 g^\alpha}{\partial (t^\alpha)^2}(x_0, t_0) = 2(\ell_1^2 + \ell_2)I$,
 $\lim_{\alpha \rightarrow \infty} \frac{\partial g^\alpha}{\partial x_i^\alpha}(x_0, t_0) = \lim_{\alpha \rightarrow \infty} \frac{\partial^2 g^\alpha}{\partial x_i^\alpha \partial x_j^\alpha}(x_0, t_0) = \lim_{\alpha \rightarrow \infty} \frac{\partial^2 g^\alpha}{\partial x_i^\alpha \partial (t^\alpha)^2}(x_0, t_0) = 0$

and all limits are uniform on (x_0, t_0) .

That is, all metrics satisfying (2.1) have coordinate systems such that the metrics A_α in these coordinate systems (at (x_0, t_0)), in the limit, depend only on ℓ_1 and ℓ_2 (up to second derivatives). Note that these limits depend just on f .

To prove the claim, take normal coordinates (x_1, \dots, x_n) for M with respect to $A|_{M \times \{t_0\}}$.

Then we have that the coordinate system (x_1, \dots, x_n, t) near (x_0, t_0) satisfies:

a.2.

- (a) $A = \sum g_{ij} dx_i dx_j + dt^2$.
- (b) $g(x_0, t_0) = I$.
- (c) There is a $C > 0$, independent of (x_0, t_0) , such that

$$\left| \frac{\partial^{k+s} g_{ij}}{\partial x_{i_1} \dots \partial x_{i_k} \partial t^s} \right| < C \quad k + s \leq 2 \quad k, s \geq 0.$$

Note that the fact that C is independent of (x_0, t_0) follows from the compactness of $M \times I$.

Now define new coordinates $(x_1^\alpha, \dots, x_n^\alpha, t^\alpha)$ as follows:

$$t^\alpha = \alpha t, \quad x_i^\alpha = f_\alpha(t_0^\alpha) x_i, \quad \text{where } t_0^\alpha = \alpha t_0.$$

Then

$$dt^\alpha = \alpha dt, \quad dx_i^\alpha = f_\alpha(t_0^\alpha) dx_i$$

and

$$\frac{\partial}{\partial t^\alpha} = \frac{1}{\alpha} \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x_i^\alpha} = \frac{1}{f_\alpha(t_0^\alpha)} \frac{\partial}{\partial x_i}.$$

We now verify that this is the coordinate system we want. (a) and (b) of the claim are direct. For (c) we make all the calculations explicitly.

We have that

$$g^\alpha = \left(\frac{f_\alpha(t^\alpha)}{f_\alpha(t_0^\alpha)} \right)^2 g.$$

First Limit.

$$\begin{aligned} \frac{\partial}{\partial t^\alpha}(g^\alpha) &= \frac{1}{\alpha} \frac{\partial}{\partial t}(g^\alpha) \\ &= \frac{1}{\alpha} \frac{\partial}{\partial t} \left[\left(\frac{f_\alpha(t^\alpha)}{f_\alpha(t_0^\alpha)} \right)^2 g \right] = \frac{1}{\alpha} \left[\frac{2\alpha f_\alpha(\alpha t) f'_\alpha(\alpha t)}{f_\alpha^2(\alpha t_0)} g + \left(\frac{f_\alpha(t^\alpha)}{f_\alpha(t_0^\alpha)} \right)^2 \frac{\partial g}{\partial t} \right]. \end{aligned}$$

At $(x, t) = (x_0, t_0)$, we have (by (b) of 2.3 and (c) of a.2)

$$\frac{\partial g^\alpha}{\partial t^\alpha}(x_0, t_0) = \frac{1}{\alpha} \left[\frac{2\alpha f'_\alpha(\alpha t_0)}{f_\alpha(\alpha t_0)} I + \frac{\partial g}{\partial t} \right] \rightarrow 2\ell_1 I.$$

Second Limit.

$$\begin{aligned} \frac{\partial^2 g^\alpha}{\partial (t^\alpha)^2} &= \frac{\partial}{\partial t^\alpha} \left\{ \frac{1}{\alpha} \left[\frac{2\alpha f_\alpha(\alpha t) f'_\alpha(\alpha t)}{f_\alpha^2(\alpha t_0)} g + \left(\frac{f_\alpha(t^\alpha)}{f_\alpha(t_0^\alpha)} \right)^2 \frac{\partial g}{\partial t} \right] \right\} \\ &= \frac{1}{\alpha^2} \left\{ 2\alpha^2 \left[\frac{(f'_\alpha(\alpha t))^2 + f_\alpha(\alpha t) f''_\alpha(\alpha t)}{f_\alpha^2(\alpha t_0)} \right] g \right. \\ &\quad \left. + 2\alpha \left[\frac{f_\alpha(\alpha t) f'_\alpha(\alpha t)}{f_\alpha^2(\alpha t_0)} \right] \frac{\partial g}{\partial t} + \left[\frac{f_\alpha(t^\alpha)}{f_\alpha(t_0^\alpha)} \right]^2 \frac{\partial^2 g}{\partial t^2} \right\}. \end{aligned}$$

At $(x, t) = (x_0, t_0)$, we have

$$\begin{aligned} \frac{\partial^2 g^\alpha}{\partial (t^\alpha)^2}(x_0, t_0) &= 2 \left\{ \left[\frac{f'_\alpha(\alpha t_0)}{f_\alpha(\alpha t_0)} \right]^2 + \left[\frac{f''_\alpha(\alpha t_0)}{f_\alpha(\alpha t_0)} \right] \right\} I \\ &\quad + \frac{2}{\alpha} \left[\frac{f'_\alpha(\alpha t_0)}{f_\alpha(\alpha t_0)} \right] \frac{\partial g}{\partial t} + \frac{1}{\alpha^2} \frac{\partial^2 g}{\partial t^2} \rightarrow 2(\ell_1^2 + \ell_2) I. \end{aligned}$$

Third Limit.

$$\frac{\partial g^\alpha}{\partial x_i^\alpha} = \frac{1}{f_\alpha(\alpha t_0)} \frac{\partial}{\partial x_i} \left[\left(\frac{f_\alpha(t^\alpha)}{f_\alpha(t_0^\alpha)} \right)^2 g \right].$$

At $(x, t) = (x_0, t_0)$, we have

$$\frac{\partial g^\alpha}{\partial x_i^\alpha}(x_0, t_0) = \frac{1}{f_\alpha(\alpha t_0)} \frac{\partial g}{\partial x_i} \rightarrow 0.$$

Fourth Limit.

$$\frac{\partial^2 g^\alpha}{\partial x_i^\alpha \partial (t^\alpha)}(x_0, t_0) = \frac{1}{f_\alpha(\alpha t_0)} \left[\frac{2f'_\alpha(\alpha t_0)}{f_\alpha(\alpha t_0)} \frac{\partial g}{\partial x_i} + \frac{\partial^2 g}{\partial t \partial x_i} \right] \rightarrow 0.$$

Fifth Limit.

$$\frac{\partial^2 g^\alpha}{\partial x_i^\alpha \partial x_j^\alpha}(x_0, t_0) = \frac{1}{f_\alpha(\alpha t_0)} \frac{\partial^2 g}{\partial x_i \partial x_j} \rightarrow 0.$$

That completes the proof of the claim.

We now complete the proof of the lemma. Let S be *any* metric on M and write $B = S + dt^2$. Let P be a 2-plane with orthonormal basis $\{v_1, v_2\}$ and write

$$v_i = \sum a_{ij} \frac{\partial}{\partial y_j^\alpha} + a_i \frac{\partial}{\partial t^\alpha},$$

where $(y_1^\alpha, \dots, y_n^\alpha, t^\alpha)$ are coordinates for B as in the claim.

Set

$$v'_i = \sum a_{ij} \frac{\partial}{\partial x_j^\alpha} + a_i \frac{\partial}{\partial t^\alpha}$$

and denote by P'' the 2-plane spanned by $\{v'_1, v'_2\}$. Now, $K_P(B_\alpha)$ and $K_{P'}(A_\alpha)$ are polynomials in g_{ij}^α and its partial derivatives (up to second order), with the same coefficients (depending on the a_{ij}). It follows that

$$\lim_{\alpha \rightarrow \infty} [K_P(B_\alpha) - K_{P'}(A_\alpha)] = 0 \quad \text{uniformly on } P.$$

Because B is a product metric, B_α is a warped metric and [3] gives the curvature

$$K_P(B_\alpha) = -\frac{f''_\alpha}{f_\alpha} \lambda^2 + \frac{S(v, w) - (f'_\alpha)^2}{f_\alpha^2} \|v\|^2$$

where $\{\lambda(\frac{\partial}{\partial t})_t + v_p, w_p\}$, with $v_p, w_p \in T_p M$ is an orthonormal basis for P and $B(v, w)$ is the curvature of B . In any case, because $f_\alpha \rightarrow \infty$ as $\alpha \rightarrow \infty$, we have that $k_0 - \frac{\epsilon}{2} < K_P(B_\alpha) < k_1 + \frac{\epsilon}{2}$, for α large enough.

So, if L is such that $|K_P(B_\alpha) - K_{P'}(A_\alpha)| < \frac{\epsilon}{2}$, for all $\alpha > L$, for all (x_0, t_0) and for all 2-planes P , the lemma follows.

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